Pearson Edexcel

Mark Scheme (Results)

January 2021

Pearson Edexcel International Advanced Level
In Further Pure Mathematics F3
Paper WFM03/01

Edexcel and BTEC Qualifications

Edexcel and BTEC qualifications are awarded by Pearson, the UK's largest awarding body. We provide a wide range of qualifications including academic, vocational, occupational and specific programmes for employers. For further information visit our qualifications websites at www.edexcel.com or www.btec.co.uk. Alternatively, you can get in touch with us using the details on our contact us page at www.edexcel.com/contactus.

Pearson: helping people progress, everywhere

Pearson aspires to be the world's leading learning company. Our aim is to help everyone progress in their lives through education. We believe in every kind of learning, for all kinds of people, wherever they are in the world. We've been involved in education for over 150 years, and by working across 70 countries, in 100 languages, we have built an international reputation for our commitment to high standards and raising achievement through innovation in education. Find out more about how we can help you and your students at: www.pearson.com/uk

January 2021
Publications Code WFM03_01_2101_MS

General Marking Guidance

- All candidates must receive the same treatment. Examiners must mark the first candidate in exactly the same way as they mark the last.
- Mark schemes should be applied positively. Candidates must be rewarded for what they have shown they can do rather than penalised for omissions.
- Examiners should mark according to the mark scheme not according to their perception of where the grade boundaries may lie.
- There is no ceiling on achievement. All marks on the mark scheme should be used appropriately.
- All the marks on the mark scheme are designed to be awarded. Examiners should always award full marks if deserved, i.e. if the answer matches the mark scheme. Examiners should also be prepared to award zero marks if the candidate's response is not worthy of credit according to the mark scheme.
- Where some judgement is required, mark schemes will provide the principles by which marks will be awarded and exemplification may be limited.
- When examiners are in doubt regarding the application of the mark scheme to a candidate's response, the team leader must be consulted.
- Crossed out work should be marked UNLESS the candidate has replaced it with an alternative response.

PEARSON EDEXCEL IAL MATHEMATICS

General Instructions for Marking

1. The total number of marks for the paper is 75

2. The Edexcel Mathematics mark schemes use the following types of marks:

- **M** marks: Method marks are awarded for 'knowing a method and attempting to apply it', unless otherwise indicated.
- **A** marks: Accuracy marks can only be awarded if the relevant method (M) marks have been earned.
- **B** marks are unconditional accuracy marks (independent of M marks)
- Marks should not be subdivided.

3. Abbreviations

These are some of the traditional marking abbreviations that will appear in the mark schemes.

- bod – benefit of doubt
- ft – follow through
- the symbol $\sqrt{}$ will be used for correct ft
- cao – correct answer only
- cso - correct solution only. There must be no errors in this part of the question to obtain this mark
- isw – ignore subsequent working
- awrt – answers which round to
- SC: special case
- o.e. – or equivalent (and appropriate)
- d… or dep – dependent
- indep – independent
- dp decimal places
- sf significant figures
- $\ast$ The answer is printed on the paper or ag- answer given
- $\llcorner$ or d… The second mark is dependent on gaining the first mark

4. All A marks are 'correct answer only' (cao), unless shown, for example, as A1ft to indicate that previous wrong working is to be followed through. After a misread however, the subsequent A marks affected are treated as A ft, but manifestly absurd answers should never be awarded A marks.

5. For misreading which does not alter the character of a question or materially simplify it, deduct two from any A or B marks gained, in that part of the question affected.

6. If a candidate makes more than one attempt at any question:
    - If all but one attempt is crossed out, mark the attempt which is NOT crossed out.
    - If either all attempts are crossed out or none are crossed out, mark all the attempts and score the highest single attempt.

7. Ignore wrong working or incorrect statements following a correct answer.

General Principles for Further Pure Mathematics Marking

(*But note that specific mark schemes may sometimes override these general principles*)

Method mark for solving 3 term quadratic:

1. Factorisation

$(x^2 + bx + c) = (x + p)(x + q)$, where $|pq| = |c|$, leading to $x = \ldots$

$(ax^2 + bx + c) = (mx + p)(nx + q)$, where $|pq| = |c|$ and $|mn| = |a|$, leading to $x = \ldots$

2. Formula

Attempt to use the correct formula (with values for a, b and c).

3. Completing the square

Solving $x^2 + bx + c = 0$: $\left(x \pm \frac{b}{2}\right)^2 \pm q \pm c = 0$, $q \neq 0$, leading to $x = \ldots$

Method marks for differentiation and integration:

1. Differentiation

Power of at least one term decreased by 1. ($x^n \to x^{n-1}$)

2. Integration

Power of at least one term increased by 1. ($x^n \to x^{n+1}$)

Question Number	Scheme	Notes	Marks
1(a)	$\pm\overrightarrow{AB}=\pm\begin{pmatrix}4\\-4\\-1\end{pmatrix},\ \pm\overrightarrow{BC}=\pm\begin{pmatrix}-1\\5\\2\end{pmatrix},\ \pm\overrightarrow{AC}=\pm\begin{pmatrix}3\\1\\1\end{pmatrix}$ Attempts any 2 of these vectors. Allow these to be written as coordinates.		M1
	E.g. $\overrightarrow{AB}\times\overrightarrow{AC}=\begin{pmatrix}4\\-4\\-1\end{pmatrix}\times\begin{pmatrix}3\\1\\1\end{pmatrix}=\begin{pmatrix}-3\\-7\\16\end{pmatrix}$	Attempts the vector product of 2 appropriate vectors. If no working is shown, look for at least 2 correct elements.	**d**M1
	$\text{Area}=\frac{1}{2}\sqrt{3^2+7^2+16^2}=\frac{1}{2}\sqrt{314}$	Correct exact area. Allow recovery from sign errors in the vector product e.g. allow following a vector product of $\pm3\mathbf{i}\pm7\mathbf{j}\pm16\mathbf{k}$	A1
	Note that a correct exact area of $\frac{1}{2}\sqrt{314}$ with no evidence of any incorrect work scores full marks		
			(3)
	Alternative 1 using cosine rule:		
	$\pm\overrightarrow{AB}=\pm\begin{pmatrix}4\\-4\\-1\end{pmatrix},\ \pm\overrightarrow{BC}=\pm\begin{pmatrix}-1\\5\\2\end{pmatrix},\ \pm\overrightarrow{AC}=\pm\begin{pmatrix}3\\1\\1\end{pmatrix}$ Attempts any 2 of these vectors		M1
	$\left\|\pm\overrightarrow{AB}\right\|=\sqrt{4^2+4^2+1^2},\ \left\|\pm\overrightarrow{BC}\right\|=\sqrt{1^2+5^2+2^2},\ \left\|\pm\overrightarrow{AC}\right\|=\sqrt{3^2+1^2+1^2}$ $\cos A=\frac{33+11-30}{2\sqrt{33}\sqrt{11}}=\frac{7\sqrt{3}}{33}$ or $\cos B=\frac{30+33-11}{2\sqrt{30}\sqrt{33}}=\frac{13\sqrt{2}}{3\sqrt{55}}$ or $\cos C=\frac{30+11-33}{2\sqrt{30}\sqrt{11}}=\frac{\sqrt{8}}{\sqrt{165}}$ (For reference A = 68.44…°, B = 34.27…°, C = 77.27…°) Attempts the magnitude of all 3 sides and attempts the cosine of one of the angles using a correctly applied cosine rule **or e.g.** $\cos A=\frac{\overrightarrow{AB}.\overrightarrow{AC}}{\sqrt{33}\sqrt{11}}=\frac{12-4-1}{\sqrt{33}\sqrt{11}}$ Finds the magnitude of 2 sides and the cosine of the included angle using a correctly applied scalar product		**d**M1
	$\text{Area}=\frac{1}{2}\sqrt{11}\sqrt{33}\sin A=\frac{1}{2}\sqrt{314}$ or $\text{Area}=\frac{1}{2}\sqrt{30}\sqrt{33}\sin B=\frac{1}{2}\sqrt{314}$ or $\text{Area}=\frac{1}{2}\sqrt{30}\sqrt{11}\sin C=\frac{1}{2}\sqrt{314}$	Correct exact area. Allow recovery from sign errors in the vectors that do not affect the calculations e.g. allow $\pm\overrightarrow{AB}=\pm4\mathbf{i}\pm4\mathbf{j}\pm\mathbf{k}$, $\pm\overrightarrow{BC}=\pm\mathbf{i}\pm5\mathbf{j}\pm2\mathbf{k}$, $\pm\overrightarrow{AC}=\pm3\mathbf{i}\pm\mathbf{j}\pm\mathbf{k}$ And allow work in decimals as long as a correct exact area is found.	A1
			(3)

Alternative 2 using scalar product:		
$\pm\overrightarrow{AB} = \pm\begin{pmatrix}4\\-4\\-1\end{pmatrix}, \pm\overrightarrow{BC} = \pm\begin{pmatrix}-1\\5\\2\end{pmatrix}, \pm\overrightarrow{AC} = \pm\begin{pmatrix}3\\1\\1\end{pmatrix}$ Attempts any 2 of these vectors		M1
A to BC is $\sqrt{AB^2 - \left(\dfrac{\overrightarrow{AB}\bullet\overrightarrow{BC}}{BC}\right)^2} = \sqrt{\dfrac{157}{15}}$ **or** B to CA is $\sqrt{BC^2 - \left(\dfrac{\overrightarrow{BC}\bullet\overrightarrow{CA}}{CA}\right)^2} = \sqrt{\dfrac{314}{11}}$ **or** C to BA is $\sqrt{AC^2 - \left(\dfrac{\overrightarrow{AC}\bullet\overrightarrow{AB}}{AB}\right)^2} = \sqrt{\dfrac{314}{33}}$ Attempts one of the altitudes of triangle ABC using a correct method		**d**M1
$\text{Area} = \dfrac{1}{2}\sqrt{30}\sqrt{\dfrac{157}{15}} = \dfrac{1}{2}\sqrt{314}$ or $\text{Area} = \dfrac{1}{2}\sqrt{11}\sqrt{\dfrac{314}{11}} = \dfrac{1}{2}\sqrt{314}$ or $\text{Area} = \dfrac{1}{2}\sqrt{33}\sqrt{\dfrac{314}{33}} = \dfrac{1}{2}\sqrt{314}$	Correct exact area. Allow work in decimals as long as a correct exact area is found.	A1
		(3)
Alternative 3 using vector products:		
$\mathbf{a}\times\mathbf{b} = \begin{pmatrix}0\\4\\-16\end{pmatrix}, \mathbf{b}\times\mathbf{c} = \begin{pmatrix}0\\-8\\20\end{pmatrix}, \mathbf{c}\times\mathbf{a} = \begin{pmatrix}-3\\-3\\12\end{pmatrix}$ Attempts these vector products		M1
$\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a} = \begin{pmatrix}-3\\-7\\16\end{pmatrix}$ Adds the appropriate vector products		**d**M1
$\text{Area} = \dfrac{1}{2}\sqrt{3^2+7^2+16^2} = \dfrac{1}{2}\sqrt{314}$	Correct exact area. Allow work in decimals as long as a correct exact area is found.	A1
		(3)

(b)	$\pm\overrightarrow{AD}=\pm\begin{pmatrix}2\\-2\\k-1\end{pmatrix},\ \pm\overrightarrow{BD}=\pm\begin{pmatrix}-2\\2\\k\end{pmatrix},\ \pm\overrightarrow{CD}=\pm\begin{pmatrix}-1\\-3\\k-2\end{pmatrix}$ Attempts one of these vectors		M1
	E.g. $\overrightarrow{AB}\times\overrightarrow{AC}.\overrightarrow{AD}=\begin{pmatrix}-3\\-7\\16\end{pmatrix}\bullet\begin{pmatrix}2\\-2\\k-1\end{pmatrix}=-6+14+16k-16$ E.g. $\overrightarrow{AB}\times\overrightarrow{AC}.\overrightarrow{BD}=\begin{pmatrix}-3\\-7\\16\end{pmatrix}\bullet\begin{pmatrix}-2\\2\\k\end{pmatrix}=6-14+16k$ E.g. $\overrightarrow{AB}\times\overrightarrow{AC}.\overrightarrow{CD}=\begin{pmatrix}-3\\-7\\16\end{pmatrix}\bullet\begin{pmatrix}-1\\-3\\k-2\end{pmatrix}=3+21+16k-32$ Attempts a suitable triple product to obtain a scalar quantity ($\frac{1}{6}$ not required here). They must be forming the triple product correctly e.g. not the magnitude of a vector. Do not be too concerned if they make slips as long as appropriate vectors are being used and a scalar quantity is obtained. **Must be an attempt at the tetrahedron *ABCD*.**		**d**M1
	Volume $=\frac{1}{3}\lvert 8k-4\rvert$	Correct volume. Must see modulus and must be 2 terms but allow equivalents e.g. $\frac{4}{3}\lvert 2k-1\rvert,\frac{1}{6}\lvert 16k-8\rvert,\frac{1}{6}\lvert 8-16k\rvert$ Award once a correct answer is seen and apply isw if necessary.	A1
			(3)
			Total 6

Question Number	Scheme	Notes	Marks
2(a)	$y=\ln(\tanh 2x)\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{\tanh 2x}\times 2\operatorname{sech}^2 2x$ **or** $y=\ln(\tanh 2x)\Rightarrow \mathrm{e}^y=\tanh 2x\Rightarrow \mathrm{e}^y\frac{\mathrm{d}y}{\mathrm{d}x}=2\operatorname{sech}^2 2x\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2\operatorname{sech}^2 2x}{\tanh 2x}$ M1: Applies the chain rule or eliminates the "ln" and differentiates implicitly to obtain to obtain $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{k\operatorname{sech}^2 2x}{\tanh 2x}$ A1: Correct derivative in any form **Note that some candidates now convert to exponential form to complete this part – see below in the alternative for scoring the final M1A1**		M1A1
	$=\frac{2\cosh 2x}{\sinh 2x}\times\frac{1}{\cosh^2 2x}=\frac{2}{\sinh 2x\cosh 2x}$	Converts to sinh2x and cosh2x correctly to obtain $\frac{k}{\sinh 2x\cosh 2x}$	M1
	$=\frac{2}{\frac{1}{2}\sinh 4x}=4\operatorname{cosech}4x$	Correct answer. Note that this is not a given answer so you can allow if e.g. a sinh becomes a sin but is then recovered but if there are any obvious errors this mark should be withheld.	A1
			(4)
	Alternative using exponentials:		
	$y=\ln(\tanh 2x)=\ln\left(\frac{\mathrm{e}^{2x}-\mathrm{e}^{-2x}}{\mathrm{e}^{2x}+\mathrm{e}^{-2x}}\right)$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{e}^{2x}+\mathrm{e}^{-2x}}{\mathrm{e}^{2x}-\mathrm{e}^{-2x}}\left(\frac{(\mathrm{e}^{2x}+\mathrm{e}^{-2x})(2\mathrm{e}^{2x}+2\mathrm{e}^{-2x})-(\mathrm{e}^{2x}-\mathrm{e}^{-2x})(2\mathrm{e}^{2x}-2\mathrm{e}^{-2x})}{(\mathrm{e}^{2x}+\mathrm{e}^{-2x})^2}\right)$ **or** $y=\ln(\tanh 2x)=\ln\left(\frac{\mathrm{e}^{2x}-\mathrm{e}^{-2x}}{\mathrm{e}^{2x}+\mathrm{e}^{-2x}}\right)=\ln(\mathrm{e}^{2x}-\mathrm{e}^{-2x})-\ln(\mathrm{e}^{2x}+\mathrm{e}^{-2x})$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2\mathrm{e}^{2x}+2\mathrm{e}^{-2x}}{\mathrm{e}^{2x}-\mathrm{e}^{-2x}}-\frac{2\mathrm{e}^{2x}-2\mathrm{e}^{-2x}}{\mathrm{e}^{2x}+\mathrm{e}^{-2x}}$ M1: Writes tanh2x correctly in terms of exponentials and applies the chain rule and quotient rule or uses the subtraction law of logs and applies the chain rule A1: Correct derivative in any form		M1A1
	$=\frac{2(\mathrm{e}^{2x}+\mathrm{e}^{-2x})^2-2(\mathrm{e}^{2x}-\mathrm{e}^{-2x})^2}{\mathrm{e}^{4x}-\mathrm{e}^{-4x}}=\frac{8}{\mathrm{e}^{4x}-\mathrm{e}^{-4x}}$ Obtains $\frac{k}{\mathrm{e}^{4x}-\mathrm{e}^{-4x}}$		M1
	$=\frac{4}{\sinh 4x}=4\operatorname{cosech}4x$	Correct answer. Note that this is not a given answer so you can allow if e.g. a sinh becomes a sin but is then recovered but if there are any obvious errors this mark should be withheld.	A1

(b) Way 1	$4\text{cosech}4x = 1 \Rightarrow \sinh 4x = 4 \Rightarrow 4x = \ln\left(4+\sqrt{4^2+1}\right)$ Changes to sinh $4x$ =… and uses the **correct** logarithmic form of arsinh to reach $4x$ = …		M1
	$x = \frac{1}{4}\ln\left(4+\sqrt{17}\right)$	This value only. Allow e.g. $x = \ln\left(4+\sqrt{17}\right)^{\frac{1}{4}}$	A1
			(2)
(b) Way 2	$4\text{cosech}4x = 1 \Rightarrow 4\times\frac{2}{e^{4x}-e^{-4x}} = 1 \Rightarrow e^{8x} - 8e^{4x} - 1 = 0$ Changes to the **correct** exponential form to reach $\frac{k}{e^{4x}-e^{-4x}}$, obtains a 3TQ in e^{4x}, solves and takes ln's to reach $4x$ = … (usual rules for solving a 3TQ do not apply as long as the above conditions are met)		M1
	$x = \frac{1}{4}\ln\left(4+\sqrt{17}\right)$	This value only. Allow e.g. $x = \ln\left(4+\sqrt{17}\right)^{\frac{1}{4}}$	A1
			(2)
			Total 6

Question Number	Scheme	Notes	Marks
3(a)	$\mathbf{A}=\begin{pmatrix}2&k&2\\2&2&k\\1&2&2\end{pmatrix}$		
	$\lvert\mathbf{A}\rvert=2(4-2k)-k(4-k)+2(4-2)=0$ $\Rightarrow k^2-8k+12=0\Rightarrow k=...$ Attempts det **A** = 0 and solves 3TQ to obtain 2 values for k Note that the usual rules for solving a 3TQ do not need to be applied as long as 2 values for k are obtained. The attempt at the determinant should be a correct expression for their row or column so allow errors only when collecting terms Note that the rule of Sarrus gives $8+k^2+8-4-4k-4k=0$		M1
	$k=2,\ 6$	Correct values.	A1
	Marks for part (a) can only be scored in their attempt at (a) and not recovered from part (b)		
			(2)
(b)	$\begin{pmatrix}2&k&2\\2&2&k\\1&2&2\end{pmatrix}\to\begin{pmatrix}4-2k&4-k&2\\2k-4&2&4-k\\k^2-4&2k-4&4-2k\end{pmatrix}\to\begin{pmatrix}4-2k&k-4&2\\4-2k&2&k-4\\k^2-4&4-2k&4-2k\end{pmatrix}$ Applies the correct method to reach at least a matrix of cofactors Should be an attempt at the minors followed by $\begin{pmatrix}+&-&+\\-&+&-\\+&-&+\end{pmatrix}$ If there is any doubt then look for at least 6 correct cofactors		M1
	$\begin{pmatrix}4-2k&k-4&2\\4-2k&2&k-4\\k^2-4&4-2k&4-2k\end{pmatrix}\to\begin{pmatrix}4-2k&4-2k&k^2-4\\k-4&2&4-2k\\2&k-4&4-2k\end{pmatrix}$ **d**M1: Attempts adjoint matrix by transposing. Dependent on previous mark. A1: Correct adjoint		**d**M1 A1
	$\mathbf{A}^{-1}=\dfrac{1}{k^2-8k+12}\begin{pmatrix}4-2k&4-2k&k^2-4\\k-4&2&4-2k\\2&k-4&4-2k\end{pmatrix}$ Fully correct inverse **or** follow through their incorrect determinant **from part (a)** where their determinant is a function of k		A1ft
	Ignore any labelling of the matrices and allow any type of brackets around the matrices		
			(4)
			Total 6

Question Number	Scheme	Notes	Marks
4	$x=4\cosh\theta \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}\theta}=4\sinh\theta$ $\Rightarrow \int \frac{1}{(x^2-16)^{\frac{3}{2}}}\,\mathrm{d}x = \int \frac{4\sinh\theta}{(16\cosh^2\theta-16)^{\frac{3}{2}}}\,\mathrm{d}\theta$ Full attempt to use the given substitution. Award for $\int \frac{1}{(x^2-16)^{\frac{3}{2}}}\,\mathrm{d}x = k\int \frac{\sinh\theta}{((4\cosh\theta)^2-16)^{\frac{3}{2}}}\,\mathrm{d}\theta$ Condone $4\cosh^2\theta$ for $(4\cosh\theta)^2$		M1
	$=\int \frac{4\sinh\theta}{(16\sinh^2\theta)^{\frac{3}{2}}}\,\mathrm{d}\theta = \int \frac{4\sinh\theta}{64\sinh^3\theta}\,\mathrm{d}\theta$ Simplifies $(16\cosh^2\theta-16)^{\frac{3}{2}}$ to the form $k\sinh^3\theta$ which may be implied by: $\int \frac{1}{(x^2-16)^{\frac{3}{2}}}\,\mathrm{d}x = k\int \frac{1}{\sinh^2\theta}\,\mathrm{d}\theta$ **Note that this is not dependent on the first M**		M1
	$=\int \frac{1}{16\sinh^2\theta}\,\mathrm{d}\theta$ Fully correct simplified integral. Allow equivalents e.g. $\frac{1}{16}\int \operatorname{cosech}^2\theta\,\mathrm{d}\theta, \int \frac{1}{(4\sinh\theta)^2}\,\mathrm{d}\theta, \int (4\sinh\theta)^{-2}\,\mathrm{d}\theta$ etc. May be implied by subsequent work.		A1
	$=\int \frac{1}{16\sinh^2\theta}\,\mathrm{d}\theta = \frac{1}{16}\int \operatorname{cosech}^2\theta\,\mathrm{d}\theta = -\frac{1}{16}\coth\theta(+c)$ Integrates to obtain $k\coth\theta$. **Depends on both previous method marks.**		**d**M1
	$=-\frac{1}{16}\frac{\cosh\theta}{\sinh\theta}+c = -\frac{1}{16}\frac{\frac{x}{4}}{\sqrt{\frac{x^2}{16}-1}}+c$ or e.g. $-\frac{1}{4}\frac{\frac{x}{4}}{\sqrt{x^2-16}}+c$ Substitutes back **correctly** for x by replacing $\cosh\theta$ with $\frac{x}{4}$ or equivalent e.g. 4cosh θ with x **and** $\sinh\theta$ with $\sqrt{\left(\frac{x}{4}\right)^2-1}$ or equivalent e.g. $4\sinh\theta$ with $\sqrt{x^2-16}$ **Depends on all previous method marks and must be fully correct work for their "$-\frac{1}{16}$"**		**d**M1
	$\frac{-x}{16\sqrt{x^2-16}}(+c)$ oe e.g. $\frac{-\frac{1}{16}x}{\sqrt{x^2-16}}(+c)$	Correct answer. Award once the correct answer is seen and apply isw if necessary. Condone the omission of "+ c"	A1
	Note that you can condone the omission of the "dθ" throughout		
			(6)
			Total 6

Question Number	Scheme	Notes	Marks
	Mark (a) and (b) together but do not credit work for (a) that is seen in (c)		
5(a)	$\begin{pmatrix}6 & -2 & -1\\ -2 & 6 & -1\\ -1 & -1 & 5\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}8x\\ 8y\\ 8z\end{pmatrix}$ or $\begin{pmatrix}-2 & -2 & -1\\ -2 & -2 & -1\\ -1 & -1 & -3\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\Rightarrow\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\ldots$ Correct method for obtaining the eigenvector		M1
	$\mathbf{i}-\mathbf{j}$	Any multiple of this vector	A1
			(2)
(b)	$\lvert\mathbf{M}-\lambda\mathbf{I}\rvert=\begin{vmatrix}6-\lambda & -2 & -1\\ -2 & 6-\lambda & -1\\ -1 & -1 & 5-\lambda\end{vmatrix}$ $\Rightarrow\underline{(6-\lambda)}\underline{\underline{((6-\lambda)(5-\lambda)-1)}}\underline{+2}\underline{\underline{(2(\lambda-5)-1)}}\ \underline{-1}\underline{\underline{(2+6-\lambda)}}$ Correct attempt at the determinant of $\mathbf{M}-\lambda\mathbf{I}$. The terms with single underlining should be correct with correct signs but allow minor slips in the brackets with double underlining. Note that the rule of Sarrus gives $(6-\lambda)(6-\lambda)(5-\lambda)-2-2-(6-\lambda)-(6-\lambda)-4(5-\lambda)$		M1
	$\Rightarrow\lambda^3-17\lambda^2+90\lambda-144=0\Rightarrow\lambda=\ldots$	Solves $\mathbf{M}-\lambda\mathbf{I}=0$ to obtain 2 different distinct real eigenvalues excluding 8	M1
	$\Rightarrow\lambda=3,6,(8)$	For 3 and 6	A1
			(3)

(c)	$(\mathbf{D}=)\begin{pmatrix}8&0&0\\0&3&0\\0&0&6\end{pmatrix}$	Correct **D** with distinct non-zero eigenvalues in any order. Follow through their non-zero 3 and 6. Ignore labelling and score for sight of the correct or correct ft matrix.	B1ft
	$\begin{pmatrix}6&-2&-1\\-2&6&-1\\-1&-1&5\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}3x\\3y\\3z\end{pmatrix}\Rightarrow\begin{pmatrix}x\\y\\z\end{pmatrix}=\ldots$ NB $\mathbf{v}_2=k\begin{pmatrix}1\\1\\1\end{pmatrix}$ **and** $\begin{pmatrix}6&-2&-1\\-2&6&-1\\-1&-1&5\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}6x\\6y\\6z\end{pmatrix}\Rightarrow\begin{pmatrix}x\\y\\z\end{pmatrix}=\ldots$ NB $\mathbf{v}_3=k\begin{pmatrix}1\\1\\-2\end{pmatrix}$ Attempts eigenvectors for their other 2 distinct eigenvalues not including 8 May use e.g. $(\mathbf{M}-\lambda\mathbf{I})\mathbf{x}=\mathbf{0}$		M1
	$(\mathbf{P}=)\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{3}}&\frac{1}{\sqrt{6}}\\-\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{3}}&\frac{1}{\sqrt{6}}\\0&\frac{1}{\sqrt{3}}&-\frac{2}{\sqrt{6}}\end{pmatrix}$ Forms a complete **P** from normalised eigenvectors using their eigenvector from part (a) and their other 2 eigenvectors formed from their other 2 different distinct eigenvalues in any order. Ignore labelling and score for forming this matrix which may be seen as part of a calculation.		M1
	$\mathbf{D}=\begin{pmatrix}8&0&0\\0&3&0\\0&0&6\end{pmatrix}$ **and** $\mathbf{P}=\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{3}}&\frac{1}{\sqrt{6}}\\-\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{3}}&\frac{1}{\sqrt{6}}\\0&\frac{1}{\sqrt{3}}&-\frac{2}{\sqrt{6}}\end{pmatrix}$ All fully correct and consistent and correctly labelled but the labelling may be implied by their working.		A1
			(4)
			Total 9

Question Number	Scheme	Notes	Marks
6(a) Way 1	$\int \frac{x^n}{\sqrt{x^2+3}}\,\mathrm{d}x = \int x^{n-1}x\left(x^2+3\right)^{-\frac{1}{2}}\,\mathrm{d}x$ **or** $\int \frac{x^n}{\sqrt{x^2+3}}\,\mathrm{d}x = \int x^{n-1}\mathrm{d}\left(x^2+3\right)^{\frac{1}{2}}$ **Applies** $x^n = x^{n-1}\times x$ to $\int \frac{x^n}{\sqrt{x^2+3}}\,\mathrm{d}x$ but may be implied by subsequent work		M1
	$\int x^{n-1}x\left(x^2+3\right)^{-\frac{1}{2}}\,\mathrm{d}x = x^{n-1}\left(x^2+3\right)^{\frac{1}{2}} - \int (n-1)x^{n-2}\left(x^2+3\right)^{\frac{1}{2}}\,\mathrm{d}x$ **d**M1: Applies integration by parts to obtain $\alpha x^{n-1}\left(x^2+3\right)^{\frac{1}{2}} - \beta\int x^{n-2}\left(x^2+3\right)^{\frac{1}{2}}\,\mathrm{d}x$ (NB α, β may be functions of n) Note that if a correct formula for parts is quoted first and parts is applied in the correct direction then we can condone slips in signs as long as the expression is of the above form. **If you are unsure – send to review.** A1: Correct expression		**d**M1A1
	$= x^{n-1}\left(x^2+3\right)^{\frac{1}{2}} - \int (n-1)x^{n-2}\left(x^2+3\right)\left(x^2+3\right)^{-\frac{1}{2}}\,\mathrm{d}x$ Applies $\left(x^2+3\right)^{\frac{1}{2}} = \left(x^2+3\right)\left(x^2+3\right)^{-\frac{1}{2}}$ **having made an attempt at integration by parts in the correct direction**		M1
	$= x^{n-1}\left(x^2+3\right)^{\frac{1}{2}} - (n-1)\int x^n\left(x^2+3\right)^{-\frac{1}{2}}\,\mathrm{d}x - 3(n-1)\int x^{n-2}\left(x^2+3\right)^{-\frac{1}{2}}\,\mathrm{d}x$ $= x^{n-1}\left(x^2+3\right)^{\frac{1}{2}} - (n-1)I_n - 3(n-1)I_{n-2}$ Splits into 2 integrals involving I_n and I_{n-2} **Depends on all the previous method marks**		**d**M1
	$\Rightarrow I_n = \frac{x^{n-1}}{n}\left(x^2+3\right)^{\frac{1}{2}} - \frac{3(n-1)}{n}I_{n-2}$ * Obtains the printed answer. You can condone the odd missing "dx" but if there are any clear errors e.g. invisible brackets that are not recovered, sign errors etc. then this mark should be withheld.		A1*
			(6)

6(a) Way 2	$$\int \frac{x^n}{\sqrt{x^2+3}}\,\mathrm{d}x = \int x^{n-2}x^2\left(x^2+3\right)^{-\frac{1}{2}}\,\mathrm{d}x$$ Applies $x^n = x^{n-2}\times x^2$	M1
	$$\int x^{n-2}x^2\left(x^2+3\right)^{-\frac{1}{2}}\,\mathrm{d}x = \int x^{n-2}\left(x^2+3-3\right)\left(x^2+3\right)^{-\frac{1}{2}}\,\mathrm{d}x$$ $$= \int x^{n-2}\left(x^2+3\right)^{\frac{1}{2}}\mathrm{d}x - \int 3x^{n-2}\left(x^2+3\right)^{-\frac{1}{2}}\mathrm{d}x$$ **d**M1: Writes x^2 as $\left(x^2+3-3\right)$ to obtain $\alpha\int x^{n-2}\left(x^2+3\right)^{\frac{1}{2}}\mathrm{d}x - \beta\int x^{n-2}\left(x^2+3\right)^{-\frac{1}{2}}\mathrm{d}x$ A1: Correct expression	**d**M1A1
	$$\int x^{n-2}\left(x^2+3\right)^{\frac{1}{2}}\mathrm{d}x = \frac{x^{n-1}}{n-1}\left(x^2+3\right)^{\frac{1}{2}} - \frac{1}{n-1}\int x^n\left(x^2+3\right)^{-\frac{1}{2}}\mathrm{d}x$$ Applies integration by parts on $\int x^{n-2}\left(x^2+3\right)^{\frac{1}{2}}\mathrm{d}x$ to obtain $$\alpha x^{n-1}\left(x^2+3\right)^{\frac{1}{2}} - \beta\int x^n\left(x^2+3\right)^{-\frac{1}{2}}\mathrm{d}x$$ Note that if a correct formula for parts is quoted first and parts is applied in the correct direction then we can condone slips in signs as long as the expression is of the above form. **If you are unsure – send to review.**	M1
	$$I_n = \frac{x^{n-1}}{n-1}\left(x^2+3\right)^{\frac{1}{2}} - \frac{1}{n-1}I_n - 3I_{n-2}$$ Brings all together and introduces I_n and I_{n-2} Depends on all the previous method marks	**d**M1
	$$\Rightarrow I_n = \frac{x^{n-1}}{n}\left(x^2+3\right)^{\frac{1}{2}} - \frac{3(n-1)}{n}I_{n-2}\ *$$ Obtains the printed answer. You can condone the odd missing "dx" but if there are any clear errors e.g. invisible brackets that are not recovered, sign errors etc. then this mark should be withheld.	A1*

(b) **Way 1**	$$I_5 = \frac{x^4}{5}\left(x^2+3\right)^{\frac{1}{2}} - \frac{12}{5}I_3$$ Applies the reduction formula once to obtain I_5 in terms of I_3 Allow slips on coefficients only	M1
	$$I_5 = \frac{x^4}{5}\left(x^2+3\right)^{\frac{1}{2}} - \frac{12}{5}\left(\frac{x^2}{3}\left(x^2+3\right)^{\frac{1}{2}} - \frac{6}{3}I_1\right)$$ Applies the reduction formula again to obtain an expression for I_5 in terms of I_1 and allow "I_1"or what they think is I_1 Allow slips on coefficients only	M1
	E.g. $$I_5 = \frac{x^4}{5}\left(x^2+3\right)^{\frac{1}{2}} - \frac{12}{5}\left(\frac{x^2}{3}\left(x^2+3\right)^{\frac{1}{2}} - \frac{6}{3}\left(x^2+3\right)^{\frac{1}{2}}\right)$$ Or e.g. $$I_5 = \frac{x^4}{5}\left(x^2+3\right)^{\frac{1}{2}} - \frac{4}{5}x^2\left(x^2+3\right)^{\frac{1}{2}} + \frac{24}{5}\left(x^2+3\right)^{\frac{1}{2}}$$ **Any** correct expression in terms of x only	A1
	$$I_5 = \frac{1}{5}\left(x^2+3\right)^{\frac{1}{2}}\left(x^4-4x^2+24\right)+k$$ Must include the "+ k" but allow other letter e.g. + c	A1
		(4)
		Total 10

(b) **Way 2**	NB $I_1 = \left(x^2+3\right)^{\frac{1}{2}}$	
	$$I_3 = \frac{x^2}{3}\left(x^2+3\right)^{\frac{1}{2}} - \frac{6}{3}I_1$$ Applies the reduction formula once to obtain I_3 in terms of I_1 and allow "I_1" or what they think is I_1 Allow slips on coefficients only	M1
	$$I_5 = \frac{x^4}{5}\left(x^2+3\right)^{\frac{1}{2}} - \frac{12}{5}\left(\frac{x^2}{3}\left(x^2+3\right)^{\frac{1}{2}} - 2I_1\right)$$ Applies the reduction formula again to obtain an expression for I_5 in terms of I_1 and allow "I_1" or what they think is I_1 Allow slips on coefficients only	M1
	E.g. $$I_5 = \frac{x^4}{5}\left(x^2+3\right)^{\frac{1}{2}} - \frac{12}{5}\left(\frac{x^2}{3}\left(x^2+3\right)^{\frac{1}{2}} - \frac{6}{3}\left(x^2+3\right)^{\frac{1}{2}}\right)$$ Or e.g. $$I_5 = \frac{x^4}{5}\left(x^2+3\right)^{\frac{1}{2}} - \frac{4}{5}x^2\left(x^2+3\right)^{\frac{1}{2}} + \frac{24}{5}\left(x^2+3\right)^{\frac{1}{2}}$$ **Any** correct expression in terms of x only	A1
	$$I_5 = \frac{1}{5}\left(x^2+3\right)^{\frac{1}{2}}\left(x^4-4x^2+24\right)+k$$ Must include the "+ k" but allow other letter e.g. + c	A1

Note that (b) is hence so must involve use of the reduction formula so a direct attempt at I_5 scores no marks. However some candidates may apply the reduction formula once as in Way 1 and then attempt I_3 directly, in which case all marks are available as the reduction formula has been used but there must be a credible attempt at I_3 to reach an expression of the required form.

See below for an example:

(b) Way 3	$I_5 = \frac{x^4}{5}\left(x^2+3\right)^{\frac{1}{2}} - \frac{12}{5}I_3$ Applies the reduction formula once to obtain I_5 in terms of I_3 Allow slips on coefficients only	M1
	$I_3 = \int \frac{x^3}{\left(x^2+3\right)^{\frac{1}{2}}}\mathrm{d}x$ $u = x^2+3 \Rightarrow I_3 = \int \frac{(u-3)^{\frac{3}{2}}}{u^{\frac{1}{2}}}\frac{\mathrm{d}u}{2(u-3)^{\frac{1}{2}}} = \frac{1}{2}\int \frac{(u-3)}{u^{\frac{1}{2}}}\mathrm{d}u = \frac{1}{3}u^{\frac{3}{2}} - 6u^{\frac{1}{2}}$ $= \frac{1}{3}\left(x^2+3\right)^{\frac{3}{2}} - 6\left(x^2+3\right)^{\frac{1}{2}}$ $I_5 = \frac{x^4}{5}\left(x^2+3\right)^{\frac{1}{2}} - \frac{12}{5}\left(\frac{1}{3}\left(x^2+3\right)^{\frac{3}{2}} - 6\left(x^2+3\right)^{\frac{1}{2}}\right)$ M1: A credible attempt to find I_3 and then expresses I_5 in terms of x A1: **Any** correct expression in terms of x only	M1A1
	$I_5 = \frac{1}{5}\left(x^2+3\right)^{\frac{1}{2}}\left(x^4-4x^2+24\right)+k$ Must include the "+ k" but allow other letter e.g. + c	A1

Question Number	Scheme	Notes	Marks
7(a)	$5\mathbf{i}+3\mathbf{j}-8\mathbf{k}$ and $2\mathbf{i}-3\mathbf{j}-6\mathbf{k}$ lie in Π_1	Identifies 2 correct vectors lying in Π_1	B1
	$\mathbf{n}=\begin{pmatrix}5\\3\\-8\end{pmatrix}\times\begin{pmatrix}2\\-3\\-6\end{pmatrix}=\begin{pmatrix}-18-24\\-\left(-30+16\right)\\-15-6\end{pmatrix}$ Attempts the vector product between 2 **correct** vectors in Π_1 If no working is shown, look for at least 2 correct elements. Or e.g. Let $\mathbf{n}=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ then $\left(a\mathbf{i}+b\mathbf{j}+c\mathbf{k}\right)\bullet\left(5\mathbf{i}+3\mathbf{j}-8\mathbf{k}\right)=0,\ \left(a\mathbf{i}+b\mathbf{j}+c\mathbf{k}\right)\bullet\left(2\mathbf{i}-3\mathbf{j}-6\mathbf{k}\right)=0$ $\Rightarrow 5a+3b-8c=0,\ 2a-3b-6c=0\Rightarrow a=2c,\ 3b=-2c\Rightarrow \mathbf{n}=...$		M1
	$=\begin{pmatrix}-42\\14\\-21\end{pmatrix}$ or e.g. $\begin{pmatrix}6\\-2\\3\end{pmatrix}$	Correct normal vector	A1
	$\left(6\mathbf{i}-2\mathbf{j}+3\mathbf{k}\right)\bullet\left(\mathbf{i}+2\mathbf{j}+\mathbf{k}\right)=...$ Attempts scalar product between their normal vector and position vector of a point in Π_1. Do not allow this mark if the "5" (or equivalent) just 'appears'. There must be some evidence for its origin e.g. **a**.**n** = … where **a** and **n** have been defined earlier. **Depends on the first method mark.**		**d**M1
	$6x-2y+3z=5$ *	Correct proof	A1*
			(5)
	Alternative 1 for (a):		
	E.g. Let equation of Π_1 be $ax+by+z=c$ 3 points on Π_1 are (1, 2, 1), (3, −1, −5) and e.g. (8, 2, −13)		B1
	$a+2b+1=c,\ 3a-b-5=c,\ 8a+2b-13=c\ \Rightarrow a=...,b=...,c=...$ Solves simultaneously for a, b and c using **correct** points		M1
	$\Rightarrow a=2, b=-\frac{2}{3}, c=\frac{5}{3}$	Correct values	A1
	$2x-\frac{2}{3}y+z=\frac{5}{3}$	Forms Cartesian equation	**d**M1
	$6x-2y+3z=5$ *	Correct proof	A1*
	Alternative 2 for (a):		
	$\left(1,2,1\right)\rightarrow 6x-2y+3z=6-4+3=5$ Shows (1, 2, 1) lies on Π_1		B1
	$\frac{x-3}{5}=\frac{y+1}{3}=\frac{z+5}{-8}\rightarrow \mathbf{r}=\begin{pmatrix}3\\-1\\-5\end{pmatrix}+\lambda\begin{pmatrix}5\\3\\-8\end{pmatrix}$ or equivalent M1: Converts l to **correct** parametric form <u>**seen as part of an attempt at this alternative**</u> allow 1 slip with one of the elements A1: Correct form		M1A1
	$6\left(3+5\lambda\right)-2\left(-1+3\lambda\right)+3\left(-5-8\lambda\right)=5$ Shows l lies in Π_1		**d**M1
	P lies in Π_1 and l lies in Π_1 so $6x-2y+3z=5$ * All correct with conclusion		A1*

(b) Way 1	$d=\frac{\lvert 6(2)-2k+3(-7)-5\rvert}{\sqrt{6^2+2^2+3^2}}$	Correct method for the shortest distance	M1
	$=\frac{1}{7}\lvert -2k-14\rvert=\frac{2}{7}\lvert k+7\rvert$ *	Correct completion	A1*
			(2)
(b) Way 2	Distance O to Π_1 is $\frac{5}{\sqrt{6^2+2^2+3^2}}$. Distance O to parallel plane containing Q is $\frac{(6\mathbf{i}-2\mathbf{j}+3\mathbf{k})\bullet(2\mathbf{i}+k\mathbf{j}-7\mathbf{k})}{\sqrt{6^2+2^2+3^2}}=\frac{-9-2k}{7}$ $d=\left\lvert\frac{5}{7}-\frac{-9-2k}{7}\right\rvert$ Correct method for the shortest distance		M1
	$=\frac{1}{7}\lvert 2k+14\rvert=\frac{2}{7}\lvert k+7\rvert$ *	Correct completion	A1*
(b) Way 3	$d=\left\lvert\frac{\overrightarrow{PQ}\bullet\mathbf{n}}{\lvert\mathbf{n}\rvert}\right\rvert=\left\lvert\frac{(\mathbf{i}+(k-2)\mathbf{j}-8\mathbf{k})\bullet(-42\mathbf{i}+14\mathbf{j}-21\mathbf{k})}{\sqrt{42^2+14^2+21^2}}\right\rvert$ Correct method for the shortest distance		M1
	$=\left\lvert\frac{-42+14k-28+168}{49}\right\rvert=\left\lvert\frac{14k+98}{49}\right\rvert=\frac{2}{7}\lvert k+7\rvert$ *	Correct completion	A1*
(c)	$\frac{2}{7}\lvert k+7\rvert=\frac{\lvert 8(2)-4k-7+3\rvert}{\sqrt{8^2+4^2+1^2}}$ Correctly attempts the distance between (2, k, −7) and Π_2 and sets equal to the result from (a). May see alternative methods here for the distance between (2, k, −7) and Π_2 e.g. finds the coordinates of a point on Π_2 e.g. R(1, 1, −7) and then finds $d=\left\lvert\frac{\overrightarrow{RQ}\bullet(8\mathbf{i}-4\mathbf{j}+\mathbf{k})}{\lvert 8\mathbf{i}-4\mathbf{j}+\mathbf{k}\rvert}\right\rvert=\left\lvert\frac{(\mathbf{i}+(k-1)\mathbf{j})\bullet(8\mathbf{i}-4\mathbf{j}+\mathbf{k})}{\sqrt{8^2+4^2+1^2}}\right\rvert=\left\lvert\frac{8-4k+4}{9}\right\rvert=\left\lvert\frac{12-4k}{9}\right\rvert$		M1
	$\frac{2}{7}(k+7)="\frac{1}{9}(12-4k)"\Rightarrow k=...$ or $\frac{2}{7}(k+7)="\frac{1}{9}(4k-12)"\Rightarrow k=...$ Attempts to solve one of these equations where their distance from Q to Π_2 is of the form ak + b where a and b are non-zero. **or** $\frac{2}{7}(k+7)="\frac{1}{9}(12-4k)"\Rightarrow\frac{4}{49}(k+7)^2="\frac{1}{81}(12-4k)^2"$ $\Rightarrow 23k^2-462k-441=0\Rightarrow k=...$ Squares both sides and attempts to solve resulting quadratic. Condone poor attempts at squaring the brackets and there is no requirement to follow the usual guidance for solving the quadratic		**d**M1
	$k=-\frac{21}{23}$ **or** $k=21$	One correct value. Must be 21 but allow equivalent exact fractions for $-\frac{21}{23}$	A1
	$k=-\frac{21}{23}$ **and** $k=21$	Both correct values. Must be 21 but allow equivalent exact fractions for $-\frac{21}{23}$ and no other values.	A1
			(4)
			Total 11

<table>
<tr><th>Question Number</th><th>Scheme</th><th>Notes</th><th>Marks</th></tr>
<tr><td rowspan="5">8(a)</td><td>$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{-2x}{1-x^2}$</td><td>Correct derivative</td><td>B1</td></tr>
<tr><td colspan="2">$1+\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2=1+\dfrac{4x^2}{\left(1-x^2\right)^2}=\dfrac{\left(1-x^2\right)^2+4x^2}{\left(1-x^2\right)^2}$ or $\dfrac{x^4-2x^2+1+4x^2}{\left(1-x^2\right)^2}$ or $\dfrac{x^4+2x^2+1}{\left(1-x^2\right)^2}$
Attempts $1+\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2$, finds common denominator and shows working in the numerator condoning sign slips only. (The denominator may be expanded)</td><td>M1</td></tr>
<tr><td>$=\dfrac{\left(1+x^2\right)^2}{\left(1-x^2\right)^2}$ or $\left(\dfrac{1+x^2}{1-x^2}\right)^2$</td><td>Fully correct expression with factorised numerator and denominator.</td><td>A1</td></tr>
<tr><td>$\displaystyle\int_{\frac{1}{2}}^{\frac{3}{4}}\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x=\int_{\frac{1}{2}}^{\frac{3}{4}}\left(\frac{1+x^2}{1-x^2}\right)\mathrm{d}x$ *</td><td>Fully correct proof with no errors and integral as printed on the question paper but allow x^2+1 for $1+x^2$ and allow
$\displaystyle\int_{\frac{1}{2}}^{\frac{3}{4}}\frac{\left(1+x^2\right)}{\left(1-x^2\right)}\,\mathrm{d}x$ or $\displaystyle\int_{\frac{1}{2}}^{\frac{3}{4}}\frac{1+x^2}{1-x^2}\,\mathrm{d}x$</td><td>A1*</td></tr>
<tr><td colspan="2"></td><td>(4)</td></tr>
</table>

<table>
<tr><td rowspan="6">(b)</td><td colspan="2">$$\frac{(x^2+1)}{(1-x^2)} = -1+\frac{2}{1-x^2} \text{ or e.g. } -1+\frac{1}{1-x}+\frac{1}{1+x}$$
Writes the improper fraction correctly</td><td>B1</td></tr>
<tr><td colspan="2">$$\int \frac{k}{1-x^2}\,\mathrm{d}x = \pm\alpha \ln\frac{1+x}{1-x}$$
Or e.g.
$$\int \frac{k}{1-x^2}\,\mathrm{d}x = \pm\alpha\ln(1+x)\pm\alpha\ln(1-x)$$
Or e.g.
$$\int \frac{k}{1-x^2}\,\mathrm{d}x = \pm\alpha\,\mathrm{artanh}\,x$$
Achieves an acceptable form for $\int \frac{k}{1-x^2}\,\mathrm{d}x$ (k constant)
(may see partial fraction approach).</td><td>M1</td></tr>
<tr><td>$$\int -1+\frac{2}{1-x^2}\,\mathrm{d}x = -x+\ln\frac{1+x}{1-x}$$</td><td>Correct integration</td><td>A1</td></tr>
<tr><td>$$\left[-x+\ln\frac{1+x}{1-x}\right]_{\frac{1}{2}}^{\frac{3}{4}} = -\frac{3}{4}+\ln 7-\left(-\frac{1}{2}+\ln 3\right)$$</td><td>Evidence that the given limits have been applied. Condone slips as long as the intention is clear.
Depends on the previous M.</td><td>dM1</td></tr>
<tr><td>$$= -\frac{1}{4}+\ln\frac{7}{3}$$</td><td>cao</td><td>A1</td></tr>
<tr><td colspan="2"></td><td>(5)</td></tr>
<tr><td></td><td colspan="2">Note that a common incorrect approach is:
$$\int \frac{(1+x^2)}{(1-x^2)}\,\mathrm{d}x = \int\left(\frac{1}{1-x^2}+\frac{x^2}{1-x^2}\right)\mathrm{d}x = \frac{1}{2}\ln\frac{1+x}{1-x}+\ldots$$
$$= \left[\frac{1}{2}\ln\frac{1+x}{1-x}+\ldots\right]_{\frac{1}{2}}^{\frac{3}{4}} = \ldots$$
If there is no attempt at $\int\left(\frac{x^2}{1-x^2}\right)\mathrm{d}x$ this will generally score B0M1A0M0A0
BUT
If there is an attempt at $\int\left(\frac{x^2}{1-x^2}\right)\mathrm{d}x$ (however poor) and evidence that the limits have been applied this will generally score B0M1A0M1A0. Condone slips with the substitution of limits as long as the intention is clear.
BUT note that attempts that consider partial fractions such as $\frac{1+x^2}{1-x^2} \equiv \frac{A}{1-x}+\frac{B}{1+x}$ will generally score no marks – if you are unsure, send to review.
Note also that $\frac{1+x^2}{1-x^2} \equiv \frac{A}{1-x}+\frac{B}{1+x}+C$ is a correct form and could score full marks.
Also, use of $\frac{(1+x^2)}{(1-x^2)} = \frac{1-x^2+2x^2}{1-x^2} = 1+\frac{2x^2}{1-x^2}$ with no attempt to deal with the $\frac{2x^2}{1-x^2}$ as an improper fraction as in the main scheme is likely to score no marks.</td><td></td></tr>
<tr><td></td><td colspan="2"></td><td>Total 9</td></tr>
</table>

Example alternative approach to integration in part (b) by substitution:

(b)	$x=\tanh\theta \Rightarrow \int \frac{(1+x^2)}{(1-x^2)}\,\mathrm{d}x = \int \frac{(1+\tanh^2\theta)}{(1-\tanh^2\theta)}\operatorname{sech}^2\theta\,\mathrm{d}\theta$ Substitutes fully		B1
	$\int \frac{(1+\tanh^2\theta)}{(1-\tanh^2\theta)}\operatorname{sech}^2\theta\,\mathrm{d}\theta = \int (1+\tanh^2\theta)\,\mathrm{d}\theta$ $= \int (2-\operatorname{sech}^2\theta)\,\mathrm{d}\theta$ Cancel and applies $\tanh^2\theta = 1-\operatorname{sech}^2\theta$		M1
	$= \int (2-\operatorname{sech}^2\theta)\,\mathrm{d}\theta = 2\theta - \tanh\theta$	Correct integration	A1
	$[2\operatorname{artanh}x - x]_{\frac{1}{2}}^{\frac{3}{4}} = 2\times\frac{1}{2}\ln\left(\frac{1+\frac{3}{4}}{1-\frac{3}{4}}\right)-\frac{3}{4}-\left(2\times\frac{1}{2}\ln\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right)-\frac{1}{2}\right)$ Evidence that the given limits have been applied. Condone slips as long as the intention is clear. **Depends on the previous M.**		**d**M1
	$= -\frac{1}{4} + \ln\frac{7}{3}$	cao	A1
			(5)

There may be other attempts at $\int \frac{1+x^2}{1-x^2}$ or $\int \left(1+\frac{2x^2}{1-x^2}\right)\mathrm{d}x$ by substitution.

Award the B mark for a correct full substitution into $\int \frac{1+x^2}{1-x^2}$ or $\int \left(1+\frac{2x^2}{1-x^2}\right)$

with $x = \mathrm{f}(\theta)$ where f is any trigonometric or hyperbolic function.

The first M mark can be scored if they reach something that is clearly directly "integrable". This will be hard to achieve for some choices like $x = \cosh\theta$
Award the first A if the integration is correct - so that requires $\int 1\,\mathrm{d}x = x$ as well if $\int \left(1+\frac{2x^2}{1-x^2}\right)$ is being attempted.

The dependent M can be awarded if there is evidence that the given limits have been applied. So score M0 if their integration has led to something that is defined outside of the limits (such as arcosh x or arcoth x). Then A1 for the correct answer.

<table>
<tr><th>Question Number</th><th>Scheme</th><th>Notes</th><th>Marks</th></tr>
<tr><td>9</td><td colspan="2">$\frac{x^2}{25}+\frac{y^2}{16}=1, \quad (5\cos\theta, 4\sin\theta)$</td><td></td></tr>
<tr><td rowspan="8">(a)</td><td>$\frac{dx}{d\theta}=-5\sin\theta,\ \frac{dy}{d\theta}=4\cos\theta$
or
$\frac{2x}{25}+\frac{2y}{16}\frac{dy}{dx}=0$ oe
or
$\frac{dy}{dx}=-\frac{4x}{25}\left(1-\frac{x^2}{25}\right)^{-\frac{1}{2}}$ oe</td><td>Correct derivatives or correct implicit differentiation or correct explicit differentiation.</td><td>B1</td></tr>
<tr><td>$\frac{dy}{dx}=\frac{4\cos\theta}{-5\sin\theta}$</td><td>Divides their derivatives correctly or substitutes and rearranges</td><td>M1</td></tr>
<tr><td>$M_N=\frac{5\sin\theta}{4\cos\theta}$</td><td>Correct perpendicular gradient rule – may be implied when they form the normal equation.</td><td>M1</td></tr>
<tr><td>$y-4\sin\theta=\frac{5\sin\theta}{4\cos\theta}(x-5\cos\theta)$</td><td>Correct straight line method (any complete method). Must use their gradient of the normal.</td><td>M1</td></tr>
<tr><td>$5x\sin\theta-4y\cos\theta=9\sin\theta\cos\theta$*
or
$9\sin\theta\cos\theta=5x\sin\theta-4y\cos\theta$*</td><td>Achieves the printed answer with no errors and allow this answer to be obtained from the previous line. Allow $5\sin\theta x$ for $5x\sin\theta$ and $4\cos\theta y$ for $4y\cos\theta$.</td><td>A1*</td></tr>
<tr><td colspan="2">Allow all marks if the gradient is seen as a function of x and y initially (even in the straight line equation) as long as this is recovered correctly.</td><td></td></tr>
<tr><td colspan="2">Solutions that do not use calculus e.g. just quoting the equation of the normal as $y-4\sin\theta=\frac{5\sin\theta}{4\cos\theta}(x-5\cos\theta)$ send to review however if they just quote e.g. $ax\sin\theta-by\sin\theta=(a^2-b^2)\sin\theta\cos\theta$ and then write down the given result this scores no marks.
But we would accept $\frac{dy}{dx}=\frac{4\cos\theta}{-5\sin\theta}$ to be quoted for a full solution.</td><td></td></tr>
<tr><td colspan="2"></td><td>(5)</td></tr>
<tr><td rowspan="3">(b)</td><td colspan="2">$b^2=a^2(1-e^2)\Rightarrow 16=25(1-e^2)\Rightarrow e=\frac{3}{5}$
F is $(ae,\ 0)=\left(5\times\frac{3}{5},\ 0\right)$
Or e.g. $"c"^2=a^2e^2=a^2-b^2=25-16\Rightarrow a^2e^2=9\Rightarrow ae=...$
Fully correct strategy for F (must be numerical so $(5e, 0)$ is M0</td><td>M1</td></tr>
<tr><td>(3, 0)</td><td>Correct coordinates. (±3, 0) scores A0</td><td>A1</td></tr>
<tr><td colspan="2"></td><td>(2)</td></tr>
</table>

(c)	$x=\frac{9}{5}\cos\theta$	Correct x coordinate (of Q)	B1
	$PF^2=\left(5\cos\theta-"3"\right)^2+\left(4\sin\theta\right)^2$ or $PF=\sqrt{\left(5\cos\theta-"3"\right)^2+\left(4\sin\theta\right)^2}$	Correct application of Pythagoras to find PF or PF^2. Their "3" should be positive but allow work in terms of e e.g. "$5e$".	M1
	$=25\cos^2\theta-30\cos\theta+9+16\sin^2\theta$ $=25\cos^2\theta-30\cos\theta+9+16\left(1-\cos^2\theta\right)$	Applies $\sin^2\theta=1-\cos^2\theta$ to obtain a quadratic expression in $\cos\theta$. If the correct identity is not seen explicitly then their working must imply that a correct identity has been used. **Depends on the previous M.**	**d**M1
	$PF=\pm\left(5-3\cos\theta\right)$ $PF^2=9\cos^2\theta-30\cos\theta+25$	Correct expression for PF or PF^2 in terms of $\cos\theta$ with terms collected.	A1
	Note that an alternative to using Pythagoras to find PF is to use $PF = ePM$ where M is the foot of the perpendicular from P to the **positive** directrix. Score M1 for $x=\frac{a}{e}=\frac{5}{3/5}\left(=\frac{25}{3}\right)$ (not $\pm\frac{25}{3}$) and **d**M1A1 for $PF=ePM=\frac{3}{5}\left(\frac{25}{3}-5\cos\theta\right)$		
	$\frac{\lvert QF\rvert}{\lvert PF\rvert}=\frac{3-\frac{9}{5}\cos\theta}{5-3\cos\theta}=\frac{3\left(1-\frac{3}{5}\cos\theta\right)}{5\left(1-\frac{3}{5}\cos\theta\right)}$ or e.g. $\frac{3}{5}\times\frac{1-\frac{3}{5}\cos\theta}{1-\frac{3}{5}\cos\theta}=\frac{3}{5}=e*$ **or e.g.** $\frac{QF^2}{PF^2}=\frac{\left(3-\frac{9}{5}\cos\theta\right)^2}{9\cos^2\theta-30\cos\theta+25}=\frac{9-\frac{54}{5}\cos\theta+\frac{81}{25}\cos^2\theta}{9\cos^2\theta-30\cos\theta+25}$ $=\frac{9\left(1-\frac{6}{5}\cos\theta+\frac{9}{25}\cos^2\theta\right)}{25\left(1-\frac{6}{5}\cos\theta+\frac{9}{25}\cos^2\theta\right)}$ or e.g. $=\frac{9}{25}\times\frac{1-\frac{6}{5}\cos\theta+\frac{9}{25}\cos^2\theta}{1-\frac{6}{5}\cos\theta+\frac{9}{25}\cos^2\theta}=\frac{9}{25}\Rightarrow\frac{QF}{PF}=\frac{3}{5}=e*$ Fully correct working including factorisation or equivalent leading to showing that $\frac{\lvert QF\rvert}{\lvert PF\rvert}=e$ with no errors and a conclusion " = e". Note that the value of e must have been seen earlier e.g. in part (b) or calculated independently somewhere in the question. Note that this mark depends on a ratio where the numerator and denominator are either both positive or both negative or modulus symbols are present throughout. This does not apply to the second case as both numerator and denominator must be positive as they are squared.		A1*
			(5)
			Total 12