Mark Scheme (Results)

Summer 2023

Pearson Edexcel International Advanced Level
In Further Pure Mathematics F3 (WFM03)
Paper 01

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General Marking Guidance

- All candidates must receive the same treatment. Examiners must mark the first candidate in exactly the same way as they mark the last.
- Mark schemes should be applied positively. Candidates must be rewarded for what they have shown they can do rather than penalised for omissions.
- Examiners should mark according to the mark scheme not according to their perception of where the grade boundaries may lie.
- There is no ceiling on achievement. All marks on the mark scheme should be used appropriately.
- All the marks on the mark scheme are designed to be awarded. Examiners should always award full marks if deserved, i.e. if the answer matches the mark scheme. Examiners should also be prepared to award zero marks if the candidate's response is not worthy of credit according to the mark scheme.
- Where some judgement is required, mark schemes will provide the principles by which marks will be awarded and exemplification may be limited.
- Crossed out work should be marked UNLESS the candidate has replaced it with an alternative response.

General Instructions for Marking

The total number of marks for the paper is 75.

Edexcel Mathematics mark schemes use the following types of marks:

'M' marks
These are marks given for a correct method or an attempt at a correct method. In Mechanics they are usually awarded for the application of some mechanical principle to produce an equation, e.g. resolving in a particular direction; taking moments about a point; applying a suvat equation; applying the conservation of momentum principle; etc.

The following criteria are usually applied to the equation.

To earn the M mark, the equation
(i) should have the correct number of terms
(ii) each term needs to be dimensionally correct

For example, in a moments equation, every term must be a 'force x distance' term or 'mass x distance', if we allow them to cancel 'g' s.

For a resolution, all terms that need to be resolved (multiplied by sin or cos) must be resolved to earn the M mark.

'M' marks are sometimes dependent (DM) on previous M marks having been earned, e.g. when two simultaneous equations have been set up by, for example, resolving in two directions and there is then an M mark for solving the equations to find a particular quantity – this M mark is often dependent on the two previous M marks having been earned.

'A' marks
These are dependent accuracy (or sometimes answer) marks and can only be awarded if the previous M mark has been earned. e.g. M0 A1 is impossible.

'B' marks
These are independent accuracy marks where there is no method (e.g. often given for a comment or for a graph).

A and B marks may be f.t. – follow through – marks.

General Abbreviations

These are some of the traditional marking abbreviations that will appear in the mark schemes:

- bod means benefit of doubt
- ft means follow through
    - the symbol $\sqrt{}$ will be used for correct ft
- cao means correct answer only
- cso means correct solution only, i.e. there must be no errors in this part of the question to obtain this mark
- isw means ignore subsequent working

- awrt means answers which round to
- SC means special case
- oe means or equivalent (and appropriate)
- dep means dependent
- indep means independent
- dp means decimal places
- sf means significant figures
- * means the answer is printed on the question paper
- means the second mark is dependent on gaining the first mark

All A marks are 'correct answer only' (cao.), unless shown, for example, as A1 ft to indicate that previous wrong working is to be followed through. After a misread however, the subsequent A marks affected are treated as A ft, but manifestly absurd answers should never be awarded A marks.

For misreading which does not alter the character of a question or materially simplify it, deduct two from any A or B marks gained, in that part of the question affected.

If a candidate makes more than one attempt at any question:

- If all but one attempt is crossed out, mark the attempt which is NOT crossed out.
- If either all attempts are crossed out or none are crossed out, mark all the attempts and score the highest single attempt.

Ignore wrong working or incorrect statements following a correct answer.

General Principles for Further Pure Mathematics Marking

(NB specific mark schemes may sometimes override these general principles)

Method mark for solving 3 term quadratic:

- Factorisation
  - $(x^2 + bx + c) = (x + p)(x + q)$, where $|pq| = |c|$, leading to $x = \ldots$
  - $(ax^2 + bx + c) = (mx + p)(nx + q)$, where $|pq| = |c|$ and $|mn| = |a|$, leading to $x = \ldots$
- Formula
  - Attempt to use the correct formula (with values for *a*, *b*and *c*).
- Completing the square
  - Solving $x^2 + bx + c = 0$: $\left(x \pm \frac{b}{2}\right)^2 \pm q \pm c = 0, \quad q \neq 0$, leading to $x = \ldots$

Method marks for differentiation and integration:

- Differentiation
  - Power of at least one term decreased by 1. ($x^n \to x^{n-1}$)
- Integration
  - Power of at least one term increased by 1. ($x^n \to x^{n+1}$)

Use of a formula

Where a method involves using a formula that has been learnt, the advice given in recent examiners' reports is that the formula should be quoted first. Normal marking procedure is as follows:

- Method mark for quoting a correct formula and attempting to use it, even if there are small errors in the substitution of values.
- Where the formula is not quoted, the method mark can be gained by implication from correct working with values but may be lost if there is any mistake in the working.

Exact answers

Examiners' reports have emphasised that where, for example, an exact answer is asked for, or working with surds is clearly required, marks will normally be lost if the candidate resorts to using rounded decimals.

Answers without working

The rubric says that these may not gain full credit. Individual mark schemes will give details of what happens in particular cases. General policy is that if it could be done "in your head", detailed working would not be required. Most candidates do show working, but there are occasional awkward cases and if the mark scheme does not cover this, please contact your team leader for advice.

Question Number	Scheme	Notes	Marks
1	$7\cosh x + 3\sinh x = 2e^x + 7 \Rightarrow$ $7\left(\frac{e^x + e^{-x}}{2}\right) + 3\left(\frac{e^x - e^{-x}}{2}\right) = 2e^x + 7$ $\left\{\frac{7}{2}e^x + \frac{7}{2}e^{-x} + \frac{3}{2}e^x - \frac{3}{2}e^{-x} = 2e^x + 7\right\}$	Substitutes at least one correct exponential form for either of the hyperbolic terms and achieves an **equation** in exponentials and constants alone	M1
	$\Rightarrow 7\left(e^{2x}+1\right) + 3\left(e^{2x}-1\right) = 4e^{2x} + 14e^x$ $\left\{\Rightarrow 5e^{2x} + 2 = 2e^{2x} + 7e^x\right\}$	Multiplies through by e^x to obtain any equation that would form a 3TQ in e^x if like terms were collected	M1
	$\Rightarrow 6e^{2x} - 14e^x + 4 = 0 \quad \left\{3e^{2x} - 7e^x + 2 = 0\right\}$	A correct three term quadratic in e^x. Could be implied by a correct root even if terms have not been collected.	A1
	$\Rightarrow \left(3e^x - 1\right)\left(e^x - 2\right) = 0 \Rightarrow e^x = ...$	Solves their 3TQ - usual rules. One correct root for their quadratic if no working. Ignore labelling of the roots even if e.g.,"x" is used.	M1
	$x = \ln 2,\ \ln\frac{1}{3}$	Both correct and simplified but do not isw if there are **other answers.** Allow $-\ln\frac{1}{2}$ for ln 2 and –ln 3 or ln 3^{-1} for $\ln\frac{1}{3}$	A1
	Answer only is 0/5		**Total 5**
	Note that it is possible to multiply through by e^{-x} to form an equation in e^{-2x}, e^{-x} and constants. Score as main scheme, e.g., $\frac{7}{2}e^x + \frac{7}{2}e^{-x} + \frac{3}{2}e^x - \frac{3}{2}e^{-x} = 2e^x + 7$ $\Rightarrow \frac{7}{2} + \frac{7}{2}e^{-2x} + \frac{3}{2} - \frac{3}{2}e^{-2x} = 2 + 7e^{-x}$ (M1) $\Rightarrow 2e^{-2x} - 7e^{-x} + 3 = 0$ (A1) $\left(2e^{-x} - 1\right)\left(e^{-x} - 3\right) = 0 \Rightarrow e^{-x} = \frac{1}{2},\ 3$ (M1) $\Rightarrow e^x = 2,\ \frac{1}{3} \Rightarrow x = \ln 2,\ \ln\frac{1}{3}$ (A1)		

Question Number	Scheme	Notes	Marks
2	Condone poor notation e.g., determinant lines used for matrix bracketing		
(a)	$\det\begin{pmatrix}2&0&0\\0&1&4\\3&-2&-3\end{pmatrix}\ \left\{=2\times(-3+8)\right\}\ =10$	Correct value for determinant, seen or stated and not just in a final answer	B1
	$\left\{\text{Minors}:\begin{pmatrix}5&-12&-3\\0&-6&-4\\0&8&2\end{pmatrix}\Rightarrow\right\}\ \text{Cofactors}:\begin{pmatrix}5&12&-3\\0&-6&4\\0&-8&2\end{pmatrix}$	Attempts the cofactor matrix with at least 6 correct elements	M1
	Inverse is $\dfrac{1}{\text{"10"}}\begin{pmatrix}5&0&0\\12&-6&-8\\-3&4&2\end{pmatrix}$ or e.g., $\begin{pmatrix}\frac{1}{2}&0&0\\\frac{6}{5}&-\frac{3}{5}&-\frac{4}{5}\\-\frac{3}{10}&\frac{2}{5}&\frac{1}{5}\end{pmatrix}$	Correct inverse but allow ft on their "10". Allow equivalent fractions/decimals. A0 if clearly obtained incorrectly	A1ft
	Work to obtain Adj(**M**) must be seen but it may be minimal, e.g., sight of the matrix of minors followed by the correct answer is acceptable. Note that B0 M1 A1 is possible.		**(3)**
(b)	$\dfrac{1}{10}\begin{pmatrix}5&0&0\\12&-6&-8\\-3&4&2\end{pmatrix}\begin{pmatrix}u\\v\\w\end{pmatrix}=\ldots$	Multiplies their $\mathbf{M}^{-1}$ by $\begin{pmatrix}u\\v\\w\end{pmatrix}$ Must use a matrix other than **M** – not just changed by application of determinant. Condone sight of $\mathbf{vM}^{-1}$ = ... but must not be a clearly incorrect multiplication method	M1
	$\begin{pmatrix}x\\y\\z\end{pmatrix}=\dfrac{1}{10}\begin{pmatrix}5u\\12u-6v-8w\\-3u+4v+2w\end{pmatrix}$ or $\begin{pmatrix}\frac{1}{2}u\\\frac{6}{5}u-\frac{3}{5}v-\frac{4}{5}w\\-\frac{3}{10}u+\frac{2}{5}v+\frac{1}{5}w\end{pmatrix}$ or $\dfrac{1}{d}\begin{pmatrix}5u\\12u-6v-8w\\-3u+4v+2w\end{pmatrix}$ or $\begin{pmatrix}\frac{5}{d}u\\\frac{12}{d}u-\frac{6}{d}v-\frac{8}{d}w\\-\frac{3}{d}u+\frac{4}{d}v+\frac{2}{d}w\end{pmatrix}$ A1ft: Two correct vector components, coordinates or equations, ft their $d\neq 0$ A1ft: All three correct ft their non-zero $d\neq 0$ Must be exact (and not rounded decimals for ft) **These ft marks are not available for an incorrect** Adj(**M**)		A1ft A1ft
			(3)
Alt Using **M**	$2x=u \qquad x=\ldots$ $y+4z=v \quad\Rightarrow\quad y=\ldots$ $3x-2y-3z=w \qquad z=\ldots$	Uses $\mathbf{M}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}u\\v\\w\end{pmatrix}$ and finds *x*, *y* and *z* as functions of *u*, *v* and *w* Condone sight of **vM** = ... but must not be a clearly incorrect multiplication method	M1
	$x=\frac{1}{2}u$ $y=\frac{6}{5}u-\frac{3}{5}v-\frac{4}{5}w$ $z=-\frac{3}{10}u+\frac{2}{5}v+\frac{1}{5}w$	A1: Two correct equations A1: All three correct Any form with terms collected	A1 A1
			(3)

Question Number	Scheme	Notes	Marks
2(c)	$3x-7y+2z=-3 \Rightarrow 3\left(\frac{1}{2}u\right)-7\left(\frac{6}{5}u-\frac{3}{5}v-\frac{4}{5}w\right)+2\left(-\frac{3}{10}u+\frac{2}{5}v+\frac{1}{5}w\right)=-3$	Substitutes their expressions into the equation for Π_1	M1
	$-15u+10v+12w=-6$	Correct **equation.** Terms in any order but constant isolated. Accept any integer multiples.	A1
			(2)
			Total 8
Alts	To gain any marks by an alternative approach, a complete attempt at a Cartesian equation for Π_2 must be made by a viable strategy e.g.,		
	general point on $3x-7y+2z=-3$ is $\left(s,\ t,\ -\frac{3}{2}s+\frac{7}{2}t-\frac{3}{2}\right)$ $\begin{pmatrix}2&0&0\\0&1&4\\3&-2&-3\end{pmatrix}\begin{pmatrix}s\\t\\-\frac{3}{2}s+\frac{7}{2}t-\frac{3}{2}\end{pmatrix} \Rightarrow \begin{matrix}u=2s\\v=-6s+15t-6\\w=\frac{15}{2}s-\frac{25}{2}t+\frac{9}{2}\end{matrix} \Rightarrow \begin{matrix}v=-3u+15t-6\\t=-\frac{2}{25}\left(w-\frac{15}{2}\left(\frac{u}{2}\right)-\frac{9}{2}\right)\end{matrix}$ $\Rightarrow v=-3u-\frac{6}{5}w+\frac{9}{2}u+\frac{27}{5}-6$ Obtains a plane equation in any Cartesian form		M1
	$\left\{v=\frac{3}{2}u-\frac{6}{5}w-\frac{3}{5} \Rightarrow\right\}$ $-15u+10v+12w=-6$	Correct **equation.** Terms in any order but constant isolated. Accept any integer multiples.	A1
			(2)
			Total 8

Question Number	Scheme	Notes	Marks
3(a) Way 1 Identities first then squares	$y=\frac{1}{2}\left(\tan x+\cot x\right)\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2}\left(\sec^2 x-\mathrm{cosec}^2 x\right)$ oe	Correct derivative. Any equivalent.	B1
	$=\frac{1}{2}\left(1+\tan^2 x-\left(1+\cot^2 x\right)\right)$ $\left\{=\frac{1}{2}\left(\tan^2 x-\cot^2 x\right)\right\}$	Applies $\sec^2 x=\pm\tan^2 x\pm 1$ and $\mathrm{cosec}^2 x=\pm\cot^2 x\pm 1$ to their derivative	M1
	$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=\frac{1}{4}\left(\tan^4 x+\cot^4 x-2\tan^2 x\cot^2 x\right)$	Squares to a 3 term expression (or 4 if middle terms uncollected) $2\tan^2 x\cot^2 x$ can be seen as 2 **Requires previous M mark.**	**d**M1
	$\left\{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=1+\frac{1}{4}\left(\tan^4 x+\cot^4 x-2\right)\right\}$ $\Rightarrow\frac{1}{4}\left(\tan^4 x+\cot^4 x+2\right)$ or $\frac{1}{4}\tan^4 x+\frac{1}{4}\cot^4 x+\frac{1}{2}$ **Not implied. Must be seen**	Adds the 1 and achieves either expression shown but allow the constant to be multiplied by $\tan^2 x\cot^2 x$ May be seen as e.g., $\frac{1}{2}\sqrt{\tan^4 x+\cot^4 x+2\tan^2 x\cot^2 x}$	A1
	$s=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x=\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\left(\tan^2 x+\cot^2 x\right)\mathrm{d}x$ * Allow $\int\frac{1}{2}\left(\tan^2 x+\cot^2 x\right)$ or $\frac{1}{2}\int\ \tan^2 x+\cot^2 x$	M1: Applies the arc length formula with their $\frac{\mathrm{d}y}{\mathrm{d}x}$ A1: Correct result achieved with no clear mathematical errors seen. Condone omission of "dx" and/or limits and **occasional** missing arguments.	M1 A1*
	Converting to sin & cos: likely to score max of 100010 unless tan & cot are convincingly recovered		**(6)**
Way 2 Squares first then identities	$y=\frac{1}{2}\left(\tan x+\cot x\right)\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2}\left(\sec^2 x-\mathrm{cosec}^2 x\right)$ oe	Correct derivative. Any equivalent.	B1
	$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=\frac{1}{4}\left(\sec^4 x+\mathrm{cosec}^4 x-2\sec^2 x\,\mathrm{cosec}^2 x\right)$	Squares a derivative **of the correct form** to obtain a 3 (or 4 if middle terms uncollected) term expression.	M1
	$=\frac{1}{4}\left(\left(1+\tan^2 x\right)^2+\left(1+\cot^2 x\right)^2-2\left(1+\tan^2 x\right)\left(1+\cot^2 x\right)\right)$ $\left\{=\frac{1}{4}\left(1+2\tan^2 x+\tan^4 x+1+2\cot^2 x+\cot^4 x-2-2\tan^2 x-2\cot^2 x-2\tan^2 x\cot^2 x\right)\right\}$	Applies $\sec^2 x=\pm\tan^2 x\pm 1$ **twice and** $\mathrm{cosec}^2 x=\pm\cot^2 x\pm 1$ **twice**. **Requires previous M mark.**	**d**M1
	$\left\{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=1+\frac{1}{4}\left(\tan^4 x+\cot^4 x-2\right)\right\}$ $\Rightarrow\frac{1}{4}\left(\tan^4 x+\cot^4 x+2\right)$ or $\frac{1}{4}\tan^4 x+\frac{1}{4}\cot^4 x+\frac{1}{2}$ **Not implied. Must be seen**	Adds the 1 and achieves either expression shown but allow the constant to be multiplied by $\tan^2 x\cot^2 x$ May be seen as e.g., $\frac{1}{2}\sqrt{\tan^4 x+\cot^4 x+2\tan^2 x\cot^2 x}$	A1
	$s=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x=\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\tan^2 x+\cot^2 x\,\mathrm{d}x$ * Allow $\int\frac{1}{2}\left(\tan^2 x+\cot^2 x\right)$ or $\frac{1}{2}\int\ \tan^2 x+\cot^2 x$	M1: Applies the arc length formula with their $\frac{\mathrm{d}y}{\mathrm{d}x}$ A1: Correct result achieved with no clear mathematical errors seen. Condone omission of "dx" and/or limits and **occasional** missing arguments.	M1 A1*
	Converting to sin & cos: likely to score max of 100010 unless tan & cot are convincingly recovered		**(6)**

Question Number	Scheme	Notes	Marks
3(b)	$\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\left(\tan^2 x+\cot^2 x\right)\,\mathrm{d}x=\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\left(\sec^2 x-1+\mathrm{cosec}^2 x-1\right)\,\mathrm{d}x$	Applies $\tan^2 x=\pm\sec^2 x\pm1$ and $\cot^2 x=\pm\mathrm{cosec}^2 x\pm1$ to the integral	M1
	Work in sin and cos must use identities (sign errors only) and lead to a result of the form below after integration condoning the absence of a term in x but allow the last M to be available following a completed attempt at integration.		
	$=\frac{1}{2}\left[\tan x-\cot x-2x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$	M1: For $\pm\sec^2 x\to\pm\tan x$ and $\pm\mathrm{cosec}^2 x\to\pm\cot x$ **Requires previous M mark.** A1: Correct integration. Limits not required.	**d**M1 A1
	$\frac{1}{2}\left(\tan\frac{\pi}{3}-\cot\frac{\pi}{3}-\frac{2\pi}{3}-\left(\tan\frac{\pi}{6}-\cot\frac{\pi}{6}-\frac{2\pi}{6}\right)\right)$ $\left\{\frac{1}{2}\left(\sqrt{3}-\frac{2\pi}{3}-\frac{\sqrt{3}}{3}-\left(\frac{\sqrt{3}}{3}-\frac{\pi}{3}-\sqrt{3}\right)\right)\right\}$	Applies the limits (see note below) following any completed attempt at integration. Allow slips provided it is a clear attempt at $\mathrm{f}\left(\frac{\pi}{3}\right)-\mathrm{f}\left(\frac{\pi}{6}\right)$	M1
	Correct answer in any exact simplified form with 2 terms e.g. $\frac{1}{2}\left(\frac{4\sqrt{3}}{3}-\frac{\pi}{3}\right)$, $\frac{2\sqrt{3}}{3}-\frac{\pi}{6}$, $\frac{2}{\sqrt{3}}-\frac{\pi}{6}$, $\frac{1}{3}\left(2\sqrt{3}-\frac{\pi}{2}\right)$, $\frac{4\sqrt{3}-\pi}{6}$		A1
	Note they may apply the limits $\frac{\pi}{4}\,\&\,\frac{\pi}{6}$ or $\frac{\pi}{3}\,\&\,\frac{\pi}{4}$ and then double the result.		**(5)**
	Just the answer or decimal answer (0.6311017628) is 0/5		**Total 11**

Question Number	Scheme	Notes	Marks
4	Allow any suitable vector notation throughout this question.		
(a)	$\begin{pmatrix}x\\y\\z\end{pmatrix}.\begin{pmatrix}-1\\3\\3\end{pmatrix}=\begin{pmatrix}2\\4\\-5\end{pmatrix}.\begin{pmatrix}-1\\3\\3\end{pmatrix}\Rightarrow...$ or $\begin{pmatrix}x\\y\\z\end{pmatrix}.\begin{pmatrix}2\\0\\-5\end{pmatrix}=\begin{pmatrix}3\\6\\-2\end{pmatrix}.\begin{pmatrix}2\\0\\-5\end{pmatrix}\Rightarrow...$ $-x+3y+3z=-5$ and $2x-5z=16$	M1: Uses **r**.**n** = **a**.**n** at least once to obtain a plane equation A1: Both correct equations. Accept in **r**.**n** = p form	M1 A1
	e.g., $x=\dfrac{16+5z}{2}$	Obtains one variable (may be written as parameter for all marks) in terms of one of the other variables	M1
	$z=\dfrac{2x-16}{5}\Rightarrow x=5+3y+3\left(\dfrac{2x-16}{5}\right)$ $\Rightarrow 5x=25+15y+6x-48\Rightarrow x=-15y+23$ $\left\{x=-15y+23=\dfrac{16+5z}{2}\right\}$	M1: Obtains the variable/parameter in terms of the third variable (or the two other variables in terms of the parameter) A1: Both correct equations	M1 A1 **(M1 on epen)**
	Alternatively, $y=\dfrac{-x+23}{15}=\dfrac{6-z}{6}$ or $z=\dfrac{2x-16}{5}=6-6y$		
	$\left\{\dfrac{x-0}{1}=\dfrac{y-\frac{23}{15}}{-\frac{1}{15}}=\dfrac{z+\frac{16}{5}}{\frac{2}{5}}\Rightarrow\right\}$ $\mathbf{r}=\begin{pmatrix}0\\\frac{23}{15}\\-\frac{16}{5}\end{pmatrix}+\lambda\begin{pmatrix}1\\-\frac{1}{15}\\\frac{2}{5}\end{pmatrix}$ M1: Attempts vector equation of line but "**r** =" may be missing. **Requires all previous M marks.** Allow numerical slips but it must be a correct method i.e., an attempt at $\Rightarrow\dfrac{x-x_1}{l}=\dfrac{y-y_1}{m}=\dfrac{z-z_1}{n}\Rightarrow\mathbf{r}=\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}+\lambda\begin{pmatrix}l\\m\\n\end{pmatrix}$ A1: Any correct equation including "**r** ="		**d**M1 A1
	Or $\left\{\dfrac{x-23}{-15}=\dfrac{y-0}{1}=\dfrac{z-6}{-6}\Rightarrow\right\}$ $\mathbf{r}=\begin{pmatrix}23\\0\\6\end{pmatrix}+\lambda\begin{pmatrix}-15\\1\\-6\end{pmatrix}$ or $\left\{\dfrac{x-8}{\frac{5}{2}}=\dfrac{y-1}{-\frac{1}{6}}=\dfrac{z-0}{1}\Rightarrow\right\}$ $\mathbf{r}=\begin{pmatrix}8\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}\frac{5}{2}\\-\frac{1}{6}\\1\end{pmatrix}$		
	Note that the line may be given in $(\mathbf{r}-\mathbf{a})\times\mathbf{b}=0$ or $\mathbf{r}\times\mathbf{b}=\mathbf{a}\times\mathbf{b}$ form		**(7)**

Question Number	Scheme	Notes	Marks
4(a) **Alt** **Finds point and vector product of normals**	$\begin{pmatrix}x\\y\\z\end{pmatrix}.\begin{pmatrix}-1\\3\\3\end{pmatrix}=\begin{pmatrix}2\\4\\-5\end{pmatrix}.\begin{pmatrix}-1\\3\\3\end{pmatrix}\Rightarrow...$ or $\begin{pmatrix}x\\y\\z\end{pmatrix}.\begin{pmatrix}2\\0\\-5\end{pmatrix}=\begin{pmatrix}3\\6\\-2\end{pmatrix}.\begin{pmatrix}2\\0\\-5\end{pmatrix}\Rightarrow...$ $-x+3y+3z=-5$ and $2x-5z=16$	M1: Uses **r**.**n** = **a**.**n** at least once to obtain a plane equation A1: Both correct equations Accept in **r**.**n** = p form	M1 A1
	e.g., $x=0\Rightarrow z=-\frac{16}{5}$	Sets one variable equal to a value and finds a value for another variable. Correct for their equations if no working.	M1
	$3y=-5-3\left(-\frac{16}{5}\right)\Rightarrow y=\frac{23}{15}\ \left\{\Rightarrow\left(0,\ \frac{23}{15},\ -\frac{16}{5}\right)\right\}$ Or e.g., $(23,\ 0,\ 6)$, $(8,1,\ 0)$ Points will have the form $(23-15\alpha,\ \alpha,\ 6-6\alpha)$	M1: Proceeds to find a value for the remaining variable. Correct for their equations if no working. A1: Correct values	M1 A1 **(M1 on epen)**
	$\begin{pmatrix}-1\\3\\3\end{pmatrix}\times\begin{pmatrix}2\\0\\-5\end{pmatrix}=...\ \Rightarrow \mathbf{r}=\begin{pmatrix}0\\\frac{23}{15}\\-\frac{16}{5}\end{pmatrix}+\lambda\begin{pmatrix}-15\\1\\-6\end{pmatrix}$ $\left\{\mathbf{r}=\begin{pmatrix}23\\0\\6\end{pmatrix}+\lambda\begin{pmatrix}-15\\1\\-6\end{pmatrix}\quad \mathbf{r}=\begin{pmatrix}8\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}-15\\1\\-6\end{pmatrix}\right\}$	**d**M1: Attempts vector product of normals (two correct components if method unclear) and forms vector equation with **point** and direction in correct places but allow for a copying error or mix up with components. **Note** that they could obtain the direction from 2 points on the line. **Requires all previous M marks.** "**r** =" may be missing. A1: Any correct equation including "**r** ="	**d**M1 A1
			(7)

Question Number	Scheme	Notes	Marks
4(b)	Note: If 0/5 allow SC 00010 for a correct volume formula seen **<u>for tetrahedron *ABCD*</u>** e.g., $\frac{1}{6}\left\lvert\overrightarrow{CD}.\left(\overrightarrow{CA}\times\overrightarrow{CB}\right)\right\rvert$ Allow with missing modulus but not vector arrows unless implied by further work.		
Way 1 STP inc. $\overrightarrow{CD}$	$\left\lvert\begin{pmatrix}-15\\1\\-6\end{pmatrix}\right\rvert=\sqrt{262}\Rightarrow\overrightarrow{CD}=\frac{5}{\sqrt{262}}\begin{pmatrix}-15\\1\\-6\end{pmatrix}$	Attempts magnitude (allow numerical slip) of their direction vector and scales correctly to length 5	M1
	Let C be the point $(8, 1, 0)$ $\overrightarrow{CA}=\begin{pmatrix}2\\4\\-5\end{pmatrix}-\begin{pmatrix}8\\1\\0\end{pmatrix}=...\left\{\begin{pmatrix}-6\\3\\-5\end{pmatrix}\right\}$ and $\overrightarrow{CB}=\begin{pmatrix}3\\6\\-2\end{pmatrix}-\begin{pmatrix}8\\1\\0\end{pmatrix}=...\left\{\begin{pmatrix}-5\\5\\-2\end{pmatrix}\right\}$	Finds vectors for **any** two edges other than *CD*. Could be implied by a distance calculation **if *C* and/or *D* defined**. This mark is not scored if either vector is in terms of a parameter unless it is assigned a value (or is eliminated appropriately) later.	M1
	$\overrightarrow{CD}.\left(\overrightarrow{CA}\times\overrightarrow{CB}\right)=\frac{5}{\sqrt{262}}\begin{pmatrix}-15\\1\\-6\end{pmatrix}.\begin{pmatrix}-6\\3\\-5\end{pmatrix}\times\begin{pmatrix}-5\\5\\-2\end{pmatrix}=...\quad\left\{=-\frac{910}{\sqrt{262}}\right\}$	Uses an appropriate scalar triple product with their vectors and finds a value. **Must not include position vectors**. Could be inexact. M0 if clear evidence of an inappropriate method	M1
	$V=\frac{1}{6}\left\lvert\overrightarrow{CD}.\left(\overrightarrow{CA}\times\overrightarrow{CB}\right)\right\rvert=...=\frac{455}{3\sqrt{262}}$ or $\frac{455\sqrt{262}}{786}$	**d**M1: Divides their STP result by 6 and obtains a positive value. Could be inexact. Modulus might not be seen. **Requires previous M mark.** A1: A correct **exact** value	**d**M1 A1
			(5)
Way 2 STP not inc. $\overrightarrow{CD}$	$\left\lvert\begin{pmatrix}-15\\1\\-6\end{pmatrix}\right\rvert=\sqrt{262}\Rightarrow\overrightarrow{CD}=\frac{5}{\sqrt{262}}\begin{pmatrix}-15\\1\\-6\end{pmatrix}$	Attempts magnitude (allow numerical slip) of their direction vector and scales correctly to length 5	M1
	Let C be the point $(8, 1, 0)$ $\overrightarrow{AC}=\begin{pmatrix}8\\1\\0\end{pmatrix}-\begin{pmatrix}2\\4\\-5\end{pmatrix}=...\left\{\begin{pmatrix}6\\-3\\5\end{pmatrix}\right\}$ and $\overrightarrow{AB}=\begin{pmatrix}3\\6\\-2\end{pmatrix}-\begin{pmatrix}2\\4\\-5\end{pmatrix}=...\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}\right\}$	Finds vectors for **any** two edges other than *CD*. Could be implied by a distance calculation **if *C* and/or *D* defined**. (See also comment for second M1 in Way 1 re use of a parameter)	M1
	$\overrightarrow{OD}=\begin{pmatrix}8\\1\\0\end{pmatrix}+\frac{5}{\sqrt{262}}\begin{pmatrix}-15\\1\\-6\end{pmatrix}\Rightarrow\overrightarrow{AD}=\begin{pmatrix}\frac{-75}{\sqrt{262}}+8\\\frac{5}{\sqrt{262}}+1\\\frac{-30}{\sqrt{262}}\end{pmatrix}-\begin{pmatrix}2\\4\\-5\end{pmatrix}=\begin{pmatrix}\frac{-75}{\sqrt{262}}+6\\\frac{5}{\sqrt{262}}-3\\\frac{-30}{\sqrt{262}}+5\end{pmatrix}$ $\Rightarrow\overrightarrow{AD}.\left(\overrightarrow{AB}\times\overrightarrow{AC}\right)=\begin{pmatrix}\frac{-75}{\sqrt{262}}+6\\\frac{5}{\sqrt{262}}-3\\\frac{-30}{\sqrt{262}}+5\end{pmatrix}.\begin{pmatrix}1\\2\\3\end{pmatrix}\times\begin{pmatrix}6\\-3\\5\end{pmatrix}=...\quad\left\{=-\frac{910}{\sqrt{262}}\right\}$	Uses an appropriate scalar triple product with their vectors and finds a value. **Must not include position vectors**. Could be inexact. M0 if clear evidence of an inappropriate method	M1
	$V=\frac{1}{6}\left\lvert\overrightarrow{AD}.\left(\overrightarrow{AB}\times\overrightarrow{AC}\right)\right\rvert=...=\frac{455}{3\sqrt{262}}$ or $\frac{455\sqrt{262}}{786}$	**d**M1: Divides their STP result by 6 and obtains a positive value. Could be inexact. Modulus might not be seen. **Requires previous M mark.** A1: A correct **exact** value	**d**M1 A1
			(5)

Question Number	Scheme	Notes	Marks
4(b) Way 3 Triangle area + perp. distance to plane & vol. of pyramid	$\left\|\begin{pmatrix}-15\\1\\-6\end{pmatrix}\right\| = \sqrt{262} \Rightarrow \overrightarrow{CD} = \frac{5}{\sqrt{262}}\begin{pmatrix}-15\\1\\-6\end{pmatrix}$	Attempts magnitude of their direction vector and scales to length 5. See note after next M below.	M1
	Let C be the point $(8, 1, 0)$ $Area\,\Delta ACD = \frac{1}{2}\left\|\overrightarrow{CD} \times \overrightarrow{CA}\right\| = \frac{1}{2}\left\|\frac{5}{\sqrt{262}}\begin{pmatrix}-15\\1\\-6\end{pmatrix} \times \begin{pmatrix}-6\\3\\-5\end{pmatrix}\right\| = \ldots \left\{= \frac{65\sqrt{19}}{2\sqrt{262}}\right\}$ Uses formula to find a value for the area of one of the faces. Must be a full method (vector product and modulus). Condone missing $\frac{1}{2}$ Any attempts by trig/Pythagoras must be complete and credible		M1
	Note: It is possible to obtain the area of a relevant triangle such as ACD by e.g., finding the length of the perpendicular distance of point A to the line and multiplying this by $\frac{1}{2}$ x 5 – in such cases allow the first M for completing a viable attempt at the height of the triangle and the second for the area (Condone missing $\frac{1}{2}$)		
	ΔACD is in Π_1 so perp. height of tetrahedron is shortest dist. of $B(3, 6, -2)$ to $-x+3y+3z=-5$: $\left\|\frac{-1\times3+3\times6+3\times(-2)+5}{\sqrt{(-1)^2+3^2+3^2}}\right\| = \ldots \left\{\frac{14}{\sqrt{19}}\right\}$	Obtains a value for the perpendicular height via formula or any credible method (examples below)	M1
	$\text{Parallel planes}: \begin{pmatrix}3\\6\\-2\end{pmatrix}.\begin{pmatrix}-1\\3\\3\end{pmatrix} = 9,\ \begin{pmatrix}2\\4\\-5\end{pmatrix}.\begin{pmatrix}-1\\3\\3\end{pmatrix} = -5 \Rightarrow \left\|\frac{-5-9}{\sqrt{(-1)^2+3^2+3^2}}\right\| = \frac{14}{\sqrt{19}}$ $\text{Projection/Resolving}: \ \overrightarrow{BA} = \begin{pmatrix}1\\2\\3\end{pmatrix} \Rightarrow \frac{\begin{pmatrix}1\\2\\3\end{pmatrix}.\begin{pmatrix}-1\\3\\3\end{pmatrix}}{\sqrt{(-1)^2+3^2+3^2}} = \frac{14}{\sqrt{19}}$		
	$V = \frac{1}{3}\times\frac{65\sqrt{19}}{2\sqrt{262}}\times\frac{14}{\sqrt{19}} = \ldots = \frac{455}{3\sqrt{262}}$ or $\frac{455\sqrt{262}}{786}$	M1: Uses $\frac{1}{3}\times\text{area}\,\Delta\times\text{perp.height}$ and obtains a positive value. $\frac{1}{2}$ must have been used for triangle area earlier unless they now use $\frac{1}{6}\times\ldots$ **Requires previous M mark**. A1: Either correct **exact** value	**d**M1 A1
			(5)
			Total 12

Question Number	Scheme	Notes	Marks
5	$\mathbf{M}=\begin{pmatrix}1 & 2 & k\\ -1 & -3 & 4\\ 2 & 6 & -8\end{pmatrix}$		
(i) & (ii) Mark the parts together	$\det\begin{pmatrix}1-\lambda & 2 & k\\ -1 & -3-\lambda & 4\\ 2 & 6 & -8-\lambda\end{pmatrix}$ $=\pm\left[(1-\lambda)\left((-3-\lambda)(-8-\lambda)-24\right)-2\left((-1)(-8-\lambda)-8\right)+k\left((-1)(6)-2(-3-\lambda)\right)\right]$	Recognisable complete attempt at $\det\left(\mathbf{M}-\lambda\mathbf{I}\right)$. May use other rows/columns. Allow $\pm$ and slips including +2 for first –2	M1
	$Sarrus \Rightarrow \pm\left[(1-\lambda)(-3-\lambda)(-8-\lambda)+(2)(4)(2)+(k)(-1)(6)-(k)(-3-\lambda)(2)-(1-\lambda)(4)(6)-(2)(-1)(-8-\lambda)\right]$		
	$=(1-\lambda)\left(\lambda^2+11\lambda\right)-2\lambda+2k\lambda$ $=-\lambda^3-10\lambda^2+9\lambda+2k\lambda$ $=\lambda\left(-\lambda^2-10\lambda+9+2k\right)$	M1: Obtains $\{\lambda\}\left(a\lambda^2+b\lambda+c+dk \text{ oe}\right)$ $a, b, c, d \neq 0$ A1: Correct expression – allow: $\pm\{\lambda\}\left(-\lambda^2-10\lambda+9+2k \text{ oe}\right)$ or $\pm\{\lambda\}\left(\lambda^2+10\lambda-9-2k \text{ oe}\right)$ Allow quadratic to be unsimplified and the marks can be implied if the initial λ has been removed	M1 A1
	{One eigenvalue is zero, if repeated then} $9+2k=0 \Rightarrow k=...$ **or** $\left\{\pm\left(-\lambda^2-10\lambda+9+2k\right)\text{has repeated roots so}\right\}$ $b^2-4ac=0 \Rightarrow \begin{cases}100-4(-1)(9+2k)=0\\ 100-4(1)(-9-2k)=0\end{cases} \Rightarrow k=...$	Attempts to set their $c+dk=0$ and solves for k **or** Considers the case of their quadratic $a\lambda^2+b\lambda+c+dk=0$ having a repeated root and uses a valid strategy to find k	M1
	Alternative approaches with $\lambda^2+10\lambda-9-2k=0$: $\left(\lambda+a\right)^2=\lambda^2+2a\lambda+a^2 \Rightarrow 2a=10 \Rightarrow -9-2k=5^2 \Rightarrow k=...$ $\text{sum of roots}=-10 \Rightarrow \text{root }=-5 \Rightarrow \text{product of roots }=(-5)^2=-9-2k \Rightarrow k=...$		
	$k=-\dfrac{9}{2}$ or $k=-17$	One correct value for k	A1
	{One eigenvalue is zero, if repeated then} $9+2k=0 \Rightarrow k=...$ **and** $\left\{\pm\left(-\lambda^2-10\lambda+9+2k\right)\text{has repeated roots so}\right\}$ $b^2-4ac=0 \Rightarrow \begin{cases}100-4(-1)(9+2k)=0\\ 100-4(1)(-9-2k)=0\end{cases} \Rightarrow k=...$	Attempts to set their $c+dk=0$ and solves for k **and** Considers the case of their quadratic $a\lambda^2+b\lambda+c+dk=0$ having a repeated root and uses a valid strategy to find k	M1
	$k=-\dfrac{9}{2}$ with eigenvalue –10 {and 0 repeated} $k=-17$ with eigenvalue –5 {repeated and 0}	Both correct values of k and the associated non-zero eigenvalues clearly assigned. No additional eigenvalues or values for k	A1
			Total 7

Question Number	Scheme	Notes	Marks
	$\frac{x^2}{16}+\frac{y^2}{9}=1 \qquad P\left(4\cos\theta,\, 3\sin\theta\right)$		
6(a)	$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{3\cos\theta}{4\sin\theta}$ or $\frac{2x}{16}+\frac{2y}{9}\frac{\mathrm{d}y}{\mathrm{d}x}=0\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{18x}{32y}$ or $\frac{x^2}{16}+\frac{y^2}{9}=1\Rightarrow y=3\left(1-\frac{x^2}{16}\right)^{\frac{1}{2}}\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3}{2}\left(1-\frac{x^2}{16}\right)^{-\frac{1}{2}}\times-\frac{2x}{16}$	Uses a correct method and finds an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ of the correct form (sign and coefficient slips only)	M1
	$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{3\cos\theta}{4\sin\theta}$ oe e.g. $-\frac{3}{4}\cot\theta$ oe	Any correct derivative in terms of θ only.	A1
	$y-3\sin\theta=-\frac{3\cos\theta}{4\sin\theta}\left(x-4\cos\theta\right)$ or or $y=-\frac{3\cos\theta}{4\sin\theta}x+c\Rightarrow 3\sin\theta=-\frac{3\cos\theta}{4\sin\theta}4\cos\theta+c$ $\Rightarrow c=\ldots\ \left\{\frac{12\sin^2\theta+12\cos^2\theta}{4\sin\theta}\right\}$	Applies correct straight line method using any gradient in terms of θ. If they use $y=mx+c$ they must substitute coordinates correctly and reach $c=\ldots$ M0 if use normal gradient	M1
	$\Rightarrow 4y\sin\theta-12\sin^2\theta=-3x\cos\theta+12\cos^2\theta$ or using $y=mx+c: y=-\frac{3\cos\theta}{4\sin\theta}x+12\Rightarrow 4y\sin\theta=-3x\cos\theta+12$ $\Rightarrow 3x\cos\theta+4y\sin\theta\left\{=12(\cos^2\theta+\sin^2\theta)\right\}=12$ M1: Multiplies through to remove fraction to obtain an equation with trig expressions in sin and cos only. Allow this mark if they go straight to the given answer from a correct equation. Can score from use of a normal gradient and/or with coordinates wrongly placed but there must have been an attempt at a line. A1*: Correct equation from correct work. $\sin^2\theta$ and $\cos^2\theta$ must be seen somewhere in the working. Accept e.g., $\sin^2\theta+\cos^2\theta=1$ seen in side-working		M1 A1*
			(5)
(b)	$y-3\sin\theta=\frac{4\sin\theta}{3\cos\theta}\left(x-4\cos\theta\right)$ oe e.g., $4x\sin\theta-3y\cos\theta=7\sin\theta\cos\theta$ or $y=\frac{4\sin\theta}{3\cos\theta}x+c$ $\Rightarrow 3\sin\theta=\frac{4\sin\theta}{3\cos\theta}4\cos\theta+c\Rightarrow c=\ldots\ \left\{\frac{-7\sin\theta\cos\theta}{3\cos\theta}\right\}$	M1: Applies correct straight line method with the negative reciprocal of their tangent gradient. If $y=mx+c$ is used coordinates must be substituted correctly and $c=\ldots$ reached A1: Any correct equation	M1 A1
			(2)

Question Number	Scheme	Notes	Marks
6(c)	A is $\left(\dfrac{4}{\cos\theta},0\right)$	Any correct x-axis intercept of the tangent. Allow e.g., $\{x=\}\dfrac{12}{3\cos\theta},\ 4\sec\theta$ Could be on a diagram or implied by midpoint	B1
	$x=0\Rightarrow y-3\sin\theta=-\dfrac{16}{3}\sin\theta\Rightarrow B$ is $\left(0,-\dfrac{7}{3}\sin\theta\right)$	Sets x = 0 in their **normal** equation (changed gradient) and finds y. Could be implied. Allow just $-\dfrac{7}{3}\sin\theta$ oe	M1
	So midpoint M of AB is $\left(\dfrac{2}{\cos\theta},-\dfrac{7}{6}\sin\theta\right)$	Any correct midpoint. Accept any equivalents and as $x=\ldots,\ y=\ldots$	A1
	$\sin^2\theta+\cos^2\theta=1\Rightarrow\left(-\dfrac{6}{7}y\right)^2+\left(\dfrac{2}{x}\right)^2=1$	Uses $\sin^2\theta+\cos^2\theta=1$ to obtain an equation in x and y only. May follow incorrect or no attempt at midpoint	M1
	$\Rightarrow\dfrac{36}{49}y^2+\dfrac{4}{x^2}=1\Rightarrow 36x^2y^2+49\times4=49x^2$ $\Rightarrow x^2\left(49-36y^2\right)=196$	**d**M1: Rearranges to the form $x^2\left(p\pm qy^2\right)=r,\quad p,\ q,\ r\in\mathbb{Z}$ **Requires all previous M marks**. A1: Correct equation	**d**M1 A1
			(6)
	Note that is possible to use e.g., $1+\tan^2\theta=\sec^2\theta$, for example: $M\left(2\sec\theta,\ \dfrac{-7\tan\theta}{6\sec\theta}\right)\Rightarrow\sec\theta=\dfrac{x}{2},\ y=\dfrac{-7\tan\theta}{3x}\Rightarrow\tan\theta=\dfrac{-3xy}{7}\Rightarrow1+\dfrac{9x^2y^2}{49}=\dfrac{x^2}{4}$ (2nd M1) $\Rightarrow1+\dfrac{9x^2y^2}{49}=\dfrac{x^2}{4}\Rightarrow196+36x^2y^2=49x^2\Rightarrow x^2\left(49-36y^2\right)=196$ (3rd M1, A1)		**Total 13**

Question Number	Scheme	Notes	Marks
7(a) **Way 1**	$I_n=\int \cosh^n 2x\ \mathrm{d}x=\int \cosh 2x \cosh^{n-1} 2x\ \mathrm{d}x$ $=\frac{1}{2}\sinh 2x\cosh^{n-1}2x-\int\frac{1}{2}\sinh 2x\times(n-1)\cosh^{n-2}2x\times 2\sinh 2x\,\mathrm{d}x$	M1: Correct split and attempts to apply parts to obtain an expression of the correct form (sign and coefficient errors only). A1: Any correct expression	M1 A1
	$\left\{=\frac{1}{2}\sinh 2x\cosh^{n-1}2x-(n-1)\int\sinh^2 2x\cosh^{n-2}2x\,\mathrm{d}x\right\}$ $=\frac{1}{2}\sinh 2x\cosh^{n-1}2x-(n-1)\int\left(\cosh^2 2x-1\right)\cosh^{n-2}2x\,\mathrm{d}x$	Applies $\sinh^2 2x=\pm\cosh^2 2x\pm 1$ **Requires previous M mark.**	**d**M1
	$\Rightarrow I_n=\frac{1}{2}\sinh 2x\cosh^{n-1}2x-(n-1)\left(I_n-I_{n-2}\right)$	Introduces I_n and I_{n-2} - not implied by given answer. **Requires previous M mark.**	**dd**M1
	$\left\{\Rightarrow nI_n=\frac{1}{2}\sinh 2x\cosh^{n-1}2x+(n-1)I_{n-2}\right\}$ $I_n=\frac{\sinh 2x\cosh^{n-1}2x}{2n}+\frac{n-1}{n}I_{n-2}$ *	Fully correct proof. Condone missing ‘dx’s. Poor bracketing must be recovered before given answer but no other errors e.g., sin for sinh, or wrong or missing arguments	A1*
	Accept e.g., $I_n=\frac{(n-1)I_{n-2}}{n}+\frac{1}{2n}\sinh 2x\cosh^{n-1}2x$		**(5)**
Way 2	$I_n=\int \cosh^n 2x\ \mathrm{d}x=\int \cosh^2 2x\cosh^{n-2}2x\ \mathrm{d}x$ $=\int\left(\sinh^2 2x+1\right)\cosh^{n-2}2x\ \mathrm{d}x$	M1: Correct split and applies $\sinh^2 2x=\pm\cosh^2 2x\pm 1$ to obtain an expression of the correct form (sign and coefficient errors only). A1: Correct expression	M1 A1
	$\left\{=\int\cosh^{n-2}2x\ \mathrm{d}x+\int\sinh^2 2x\cosh^{n-2}2x\ \mathrm{d}x\right\}$ $\int\sinh^2 2x\cosh^{n-2}2x\ \mathrm{d}x\left\{=\int\sinh 2x\cosh^{n-2}2x\sinh 2x\ \mathrm{d}x\right\}$ $=\frac{1}{2\left(n-1\right)}\sinh 2x\cosh^{n-1}2x-\frac{1}{n-1}\int\cosh^n 2x\ \mathrm{d}x$	Attempts to apply parts to obtain an expression of the correct form for $\int\sinh^2 2x\cosh^{n-2}2x\ \mathrm{d}x$ **Requires previous M mark.**	**d**M1
	$\Rightarrow I_n=I_{n-2}+\frac{1}{2\left(n-1\right)}\sinh 2x\cosh^{n-1}2x-\frac{1}{n-1}I_n$	Introduces I_n and I_{n-2} - not implied by given answer. **Requires previous M mark.**	**dd**M1
	$\left\{\Rightarrow\left(n-1\right)I_n=\frac{1}{2}\sinh 2x\cosh^{n-1}2x+(n-1)I_{n-2}-I_n\right\}$ $I_n=\frac{\sinh 2x\cosh^{n-1}2x}{2n}+\frac{n-1}{n}I_{n-2}$ *	Fully correct proof. Condone missing ‘dx’s. Poor bracketing must be recovered before given answer but no other errors e.g., sin for sinh, or wrong or missing arguments	A1*
	Accept e.g., $I_n=\frac{(n-1)I_{n-2}}{n}+\frac{1}{2n}\sinh 2x\cosh^{n-1}2x$		**(5)**

Question Number	Scheme	Notes	Marks
7(b)	$\left(1+\cosh 2x\right)^3 = 1+3\cosh 2x+3\cosh^2 2x+\cosh^3 2x$ Correct expansion. Could be implied e.g. by $x+3I_1+3I_2+I_3$ and allow if correct but terms are not collected. Condone if partially or completely in "x" provided terms <u>are</u> collected		B1
	$\int \cosh^2 2x \,\mathrm{d}x$ or $I_2 = \frac{1}{4}\sinh 2x\cosh 2x + \frac{1}{2}I_0$ or $\int \cosh^3 2x \,\mathrm{d}x$ or $I_3 = \frac{1}{6}\sinh 2x\cosh^2 2x + \frac{2}{3}I_1$	Completes an attempt to apply the reduction formula for I_2 **or** I_3. May be slips but must get two terms. May be seen with I_0 / I_1 attempted and/or embedded in expression for $\int \left(1+\cosh 2x\right)^3 \mathrm{d}x$	M1
	$I_0 = x \qquad I_1 = \frac{1}{2}\sinh 2x$ $\int \left(1+\cosh 2x\right)^3 \mathrm{d}x = \int \left(1+3\cosh 2x\right)\mathrm{d}x + 3I_2 + I_3 =$ $x+\frac{3}{2}\sinh 2x+\frac{3}{4}\sinh 2x\cosh 2x+\frac{3}{2}x+\frac{1}{6}\sinh 2x\cosh^2 2x+\frac{1}{3}\sinh 2x\left(+c\right)$	$I_0 = x$ **and** $I_1 = \pm k\sinh 2x$ (condone I_1 from formula) **and** $\int \left(1+3\cosh 2x\right)\mathrm{d}x \to x \pm q\sinh 2x$ **and** uses the above to obtain an expression for $\int \left(1+\cosh 2x\right)^3 \mathrm{d}x$ **Requires previous M mark.**	**d**M1
	Note: **One** of I_2 and I_3 may be attempted directly – if so correct identities must be used and an expression of a correct form obtained. Examples: $I_2 = \int \cosh^2 2x \,\mathrm{d}x = \int \left(\frac{1}{2}\cosh 4x + \frac{1}{2}\right)\mathrm{d}x = \frac{1}{8}\sinh 4x + \frac{x}{2}$ $\Rightarrow x+\frac{3}{2}\sinh 2x+\frac{3}{8}\sinh 4x+\frac{3}{2}x+\frac{1}{6}\sinh 2x\cosh^2 2x+\frac{1}{3}\sinh 2x\left(+c\right)$ $I_3 = \int \cosh^3 2x \,\mathrm{d}x = \int \cosh 2x\left(\sinh^2 2x+1\right)\mathrm{d}x = \frac{1}{6}\sinh^3 2x + \frac{1}{2}\sinh 2x$ $\Rightarrow x+\frac{3}{2}\sinh 2x+\frac{3}{4}\sinh 2x\cosh 2x+\frac{3}{2}x+\frac{1}{6}\sinh^3 2x+\frac{1}{2}\sinh 2x\left(+c\right)$ If exponential definitions are used they must be correct.		
	$=\frac{5}{2}x+\frac{11}{6}\sinh 2x+\frac{3}{4}\sinh 2x\cosh 2x+\frac{1}{6}\sinh 2x\cosh^2 2x\left(+c\right)$	Correct answer. Award when a correct expression with collected like terms is seen.	A1
	I_2 attempted directly $\Rightarrow \frac{5}{2}x+\frac{11}{6}\sinh 2x+\frac{3}{8}\sinh 4x+\frac{1}{6}\sinh 2x\cosh^2 2x\left(+c\right)$ I_3 attempted directly $\Rightarrow \frac{5}{2}x+2\sinh 2x+\frac{3}{4}\sinh 2x\cosh 2x+\frac{1}{6}\sinh^3 2x\left(+c\right)$		**(4)**
	If identities are used before a correct answer is seen with like terms collected then the work must be correct		**Total 9**

Question Number	Scheme	Notes	Marks
8(a)	$\left\{\dfrac{\mathrm{d}y}{\mathrm{d}x}=\right\}\text{arcosh}\,5x+\dfrac{ax}{\sqrt{bx^2-1}}$ or $\text{arcosh}\,5x+\dfrac{cx}{\sqrt{x^2-d}}$ (M1) $\Rightarrow \text{arcosh}\left(5x\right)+\dfrac{5x}{\sqrt{25x^2-1}}$ (A1) M1: Differentiates to obtain expression of the correct form $a,\ b,\ c,\ d\neq 0$ A1: Correct differentiation. Any equivalent form.		M1 A1
			(2)
(b)	$\dfrac{\mathrm{d}}{\mathrm{d}x}\left(x\text{arcosh}\left(5x\right)\right)=\text{arcosh}\left(5x\right)+\text{"}\dfrac{5x}{\sqrt{25x^2-1}}\text{"}\Rightarrow\int\text{arcosh}\left(5x\right)\mathrm{d}x=x\,\text{arcosh}\left(5x\right)-\int\text{"}\dfrac{5x}{\sqrt{25x^2-1}}\text{"}\mathrm{d}x$ M1: Rearranges **their** answer to (a) correctly and integrates or uses the correct formula to apply parts to $1\times\text{arcosh}\,5x$ to obtain the above.		M1
	$\int\text{arcosh}\left(5x\right)\mathrm{d}x=x\,\text{arcosh}\left(5x\right)-\int\dfrac{5x}{\sqrt{25x^2-1}}\mathrm{d}x$ A1: **Correct** expression – but **see note below on limited ft**		A1 **(limited ft)**
	$=x\,\text{arcosh}\left(5x\right)-\dfrac{1}{5}\left(25x^2-1\right)^{\frac{1}{2}}\ \ (+c)$	M1: $\int\dfrac{Ax}{\sqrt{Bx^2-1}}\mathrm{d}x\to C\left(Bx^2-1\right)^{\frac{1}{2}}$ A1: Fully correct expression with $x$arcosh($5x$) - see note below for limited ft	M1 A1 **(limited ft)**
	Note: Substitutions: $u=5x\Rightarrow\left(u^2-1\right)^{\frac{1}{2}}\Rightarrow\left[\dfrac{1}{5}\sqrt{u^2-1}\right]_{\frac{5}{4}}^{3}\quad u=25x^2-1\Rightarrow\left[\dfrac{1}{5}\sqrt{u}\right]_{\frac{9}{16}}^{8}$ M1: Correct form A1: Fully correct expression with $x$arcosh($5x$)		
	A limited ft for **<u>one</u>** of the errors in (a) shown below applies for the first two A marks. **However also allow the following if this error occurs in part (b)** which is most likely to come from not rearranging and effectively restarting by using parts. Note that substitutions could be used. $a=1\Rightarrow x\,\text{arcosh}\left(5x\right)-\int\dfrac{x}{\sqrt{25x^2-1}}\mathrm{d}x\Rightarrow x\,\text{arcosh}\left(5x\right)-\dfrac{1}{25}\left(25x^2-1\right)^{\frac{1}{2}}\ \ (+c)$ $b=5\Rightarrow x\,\text{arcosh}\left(5x\right)-\int\dfrac{5x}{\sqrt{5x^2-1}}\mathrm{d}x\Rightarrow x\,\text{arcosh}\left(5x\right)-\left(5x^2-1\right)^{\frac{1}{2}}\ \ (+c)$ $a=-5\Rightarrow x\,\text{arcosh}\left(5x\right)+\int\dfrac{5x}{\sqrt{25x^2-1}}\mathrm{d}x\Rightarrow x\,\text{arcosh}\left(5x\right)+\dfrac{1}{5}\left(25x^2-1\right)^{\frac{1}{2}}\ \ (+c)$		
	$\int_{\frac{1}{4}}^{\frac{3}{5}}\text{arcosh}\,5x\,\mathrm{d}x=\dfrac{3}{5}\text{arcosh}\left(3\right)-\dfrac{1}{5}\sqrt{25\times\dfrac{9}{25}-1}-\left(\dfrac{1}{4}\text{arcosh}\left(\dfrac{5}{4}\right)-\dfrac{1}{5}\sqrt{25\times\dfrac{1}{16}-1}\right)$ Applies appropriate limits (note substitutions above) with subtraction the right way round seen to obtain an expression of the form $x\,\text{arcosh}\left(5x\right)\pm\mathrm{f}\left(x\right)$ **where f(*x*) has come from integration**		M1
	$=\dfrac{3}{5}\text{arcosh}(3)-\dfrac{2\sqrt{2}}{5}-\dfrac{1}{4}\text{arcosh}\left(\dfrac{5}{4}\right)+\dfrac{3}{20}$	Correct answer seen in any form. **Must not follow clearly incorrect work.**	A1
	$\text{arcosh}3=\ln\left(3+\sqrt{3^2-1^2}\right)$ or $\text{arcosh}\left(\dfrac{5}{4}\right)=\ln\left(\dfrac{5}{4}+\sqrt{\left(\dfrac{5}{4}\right)^2-1^2}\right)$ $\left\{\Rightarrow\dfrac{3}{5}\ln\left(3+\sqrt{8}\right)-\dfrac{2\sqrt{2}}{5}-\dfrac{1}{4}\ln 2+\dfrac{3}{20}\right\}$	Converts $\text{arcosh}\left(3\right)$ or $\text{arcosh}\left(\frac{5}{4}\right)$ to any correct log form. Independent mark but must have obtained $x\,\text{arcosh}\left(5x\right)\pm\mathrm{f}\left(x\right)$ **where f(*x*) has come from integration**	M1
	$=\dfrac{3}{20}-\dfrac{2\sqrt{2}}{5}+\ln\left(3+2\sqrt{2}\right)^{\frac{3}{5}}-\dfrac{1}{4}\ln 2$ **Must not follow clearly incorrect work.**	Correct answer. Terms in any order but otherwise written as shown. Allow values for p, q, r & k	A1
			(8)
			Total 10
			PAPER TOTAL: 75

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